Linear Gaussian Estimation

assumption: discrete, linear, time varying

- motion model: (random, input, deterministic, random, measurement noise)

$$x_k = A_{k-1}x_{k-1} + v_k + w_k$$
$$y_k = C_k x_k + n_k$$

$x_0 \in \mathbb{R}^N \sim \mathcal{N}(\check{x}_0, \check{P}_0)$

$w_k \in \mathbb{R}^N \sim \mathcal{N}(0, Q_k)$

$n_k \in \mathbb{R}^M \sim \mathcal{N}(0, R_k)$

- batch linear-Gaussian estimation problem
  - Bayesian
  - Maximum a Posteriori

Maximum a Posteriori

- $\hat{x} = \arg\max_x P(x \mid v, y)$

$x = x_{0:K} = (x_0, x_1, \dots x_K)$

$v = (\check{x}_0, v_{1:K}) = (\check{x}_0, v_1, \dots v_K)$ } for prior

$y = y_{0:K} = (y_0, \dots y_K)$ } for posterior

- Bayes' rule:

$$\hat{x} = \arg\max_x P(x|v,y)$$ (does not affect y)

$$= \arg\max_x \frac{P(y|x,v)P(x|v)}{P(y|v)}$$ (does not affect x)

$$= \arg\max_x P(y|x)P(x|v)$$

$P(A|B) = \frac{P(A\cap B)}{P(B)}$

$P(B|A) = \frac{P(A\cap B)}{P(A)}$

$\therefore P(A\cap B) = P(B|A)P(A)$

$\therefore P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

$P(A|BC) = \frac{P(A\cap B\cap C)}{P(B\cap C)}$

$P(C|AB) = \frac{P(A\cap B\cap C)}{P(A\cap B)}$

$\therefore P(A\cap B\cap C) = P(A\cap B)P(C|A\cap B) = P(A|B)P(B)P(C|A\cap B)$

$\therefore P(A|BC) = \frac{P(A|B)P(B)P(C|A\cap B)}{P(B)P(C|B)} = \frac{P(A|B)P(C|AB)}{P(C|B)}$

- assume each w_k, n_k are NOT correlated:

$$P(y|x) = \prod_{k=0}^{K} P(y_k|x_k)$$

- Bayes' Rule:

$$P(x|v) = P(x_0|\check{x}_0)\prod_{k=0}^{K} P(x_k|x_{k-1}, v_k)$$

$\therefore P(x_0|\check{x}_0)$

$$= \frac{1}{\sqrt{(2\pi)^N \det \check{P}_0}} \times \exp\left(-\frac{1}{2}(x_0-\check{x}_0)^T\check{P}_0^{-1}(x_0-\check{x}_0)\right)$$

$\therefore P(x_k|x_{k-1}, v_k)$

$$= \frac{1}{\sqrt{(2\pi)^N \det Q_k}} \times \exp\left(-\frac{1}{2}(x_k-(A_{k-1}x_{k-1}+v_k))^T Q_k^{-1}(x_k-(A_{k-1}x_{k-1}+v_k))\right)$$

$\therefore P(y_k|x_k)$

$$= \frac{1}{\sqrt{(2\pi)^M \det R_k}} \times \exp\left(-\frac{1}{2}(y_k - C_k x_k)^T R_k^{-1}(y_k - C_k x_k)\right)$$

- logarithm

$\ln(P(y|x)P(x|v))$

$$= \ln P(x_0|\check{x}_0) + \sum_{k=1}^{K}\ln P(x_k|x_{k-1}, v_k) + \sum_{k=1}^{K}\ln P(y_k|x_k)$$

where

$$\ln P(x_0|\check{x}_0) = -\frac{1}{2}(x_0-\check{x}_0)^T\check{P}_0^{-1}(x_0-\check{x}_0) + \cancel{-\frac{1}{2}\ln((2\pi)^N \det \check{P}_0)}$$

$$\ln P(x_k|x_{k-1}, v_k) = -\frac{1}{2}(x_k-A_{k-1}x_{k-1}-v_k)^T Q_k^{-1}(x_k-A_{k-1}x_{k-1}-v_k) + \cancel{-\frac{1}{2}\ln((2\pi)^N \det Q_k)}$$

$$\ln P(y_k|x_k) = -\frac{1}{2}(y_k-C_kx_k)^T R_k^{-1}(y_k-C_kx_k) + \cancel{-\frac{1}{2}\ln((2\pi)^M \det R_k)}$$

- J, cost

$$J_{v,k}(x) = \begin{cases} \frac{1}{2}(x_0-\check{x}_0)^T\check{P}_0^{-1}(x_0-\check{x}_0) \\ \frac{1}{2}(x_k-A_{k-1}x_{k-1}-v_k)^T Q_k^{-1}(x_k-A_{k-1}x_{k-1}-v_k) \end{cases}$$

$$J_{y,k}(x) = \frac{1}{2}(y_k-C_kx_k)^T R_k^{-1}(y_k-C_kx_k)$$

Mahalanobis distance

$$J(x) = \sum_{k=0}^{K}(J_{v,k}(x) + J_{y,k}(x))$$

$$\therefore x = \arg\min_x J(x)$$

$k=$ x_1 x_{10}

- from previous

$$J_{v,k}(x) = \begin{cases} \frac{1}{2}(x_0-\check{x}_0)^T\check{P}_0^{-1}(x_0-\check{x}_0) \\ \frac{1}{2}(x_k-A_{k-1}x_{k-1}-v_k)^T Q_k^{-1}(x_k-A_{k-1}x_{k-1}-v_k) \end{cases}$$

$$J_{y,k}(x) = \frac{1}{2}(y_k-C_kx_k)^T R_k^{-1}(y_k-C_kx_k)$$

let

$$z = \begin{bmatrix} \check{x}_0 \\ v_1 \\ \vdots \\ v_K \\ y_0 \\ y_1 \\ \vdots \\ y_K \end{bmatrix}, \quad x = \begin{bmatrix} x_0 \\ \vdots \\ x_K \end{bmatrix}$$

&

$$H = \begin{bmatrix} 1 & & & \\ -A_0 & 1 & & \\ & \ddots & \ddots & \\ & & -A_{K-1} & 1 \\ C_0 & & & \\ & C_1 & & \\ & & \ddots & \\ & & & C_K \end{bmatrix}$$

$$W = \begin{bmatrix} \check{P}_0 & & & & & \\ & Q_1 & & & & \\ & & \ddots & & & \\ & & & Q_K & & \\ & & & & R_0 & \\ & & & & & R_1 \ddots R_K \end{bmatrix}$$

Hx, z, $-A_1 x_0 + x_1 = B_u$, $x_1 =$

$$\therefore J(x) = \frac{1}{2}(z-Hx)^T W^{-1}(z-Hx)$$

$$P(z|x) = \eta \exp\left(-\frac{1}{2}(z-Hx)^T W^{-1}(z-Hx)\right)$$

η: normalization constant

$$\therefore \left.\frac{\partial J(x)}{\partial x^T}\right|_{\hat{x}} = -H^T W^{-1}(z - H\hat{x}) = 0$$

$$\Rightarrow (H^T W^{-1} H)\hat{x} = H^T W^{-1} z$$

- on $(H^T W^{-1} H)\hat{x} = H^T W^{-1} z$
- unique solution:

$$\hat{x} = \underbrace{(H^T W^{-1} H)^{-1}}_{\text{need to be invertible}} H^T W^{-1} z$$

$$w\ \hat{x} \in \mathbb{R}^{K+1}$$

$$\text{rank}(H^T W^{-1} H) = K+1$$

$$\Rightarrow \text{rank}(H^T H) = \text{rank}(H^T)$$

- $\text{rank}(H^T)$: we hv $\check{x}_0$ / we don't hv $\check{x}_0$

✓ - we hv $\check{x}_0$

rank H^T

$$= \text{rank}\begin{bmatrix} 1 & -A_0^T & & & C_0^T & & & \\ & 1 & -A_1^T & & & C_1^T & & \\ & & \ddots & -A_{K-1}^T & & & C_2^T & \\ & & & 1 & & & & C_K^T \end{bmatrix}$$

$\Rightarrow$ full rank. $\therefore$ ($Q_k \succ 0, R_k \succ 0 \Rightarrow$ hv sol.)

✓ - we don't hv $\check{x}_0$

rank H^T

$$= \text{rank}\begin{bmatrix} -A_0^T & & & & C_0^T & & & \\ 1 & -A_1^T & & & & C_1^T & & \\ & 1 & \ddots & & & & C_2^T & \\ & & & -A_{K-1}^T & & & & \\ & & & 1 & & & & C_K^T \end{bmatrix}$$

$$\Rightarrow \text{rank}\begin{bmatrix} 1 & -A_1^T & & & C_1^T & & \\ & 1 & \ddots & & & C_2^T & \\ & & & -A_{K-1}^T & & & \\ & & & 1 & & & C_K^T \\ -A_0^T & & & & C_0^T & & \end{bmatrix}$$

$$\Rightarrow \text{rank}\begin{bmatrix} 1 & -A_1^T & & & C_1^T & & \\ & 1 & \ddots & & & C_2^T & \\ & & & -A_{K-1}^T & & & C_K^T \\ & & & & C_0^T\ A_0^TC_1^T\ A_0^TA_1^TC_2^T \cdots A_0^T\cdots A_{K-1}^TC_K^T & & \end{bmatrix}$$

make sure this full R

- $\text{rank}[C_0^T\ A_0^TC_1^T\ A_0^TA_1^TC_2^T \cdots A_0^T\cdots A_{K-1}^TC_K^T] = N$

from Cayley-Hamilton theorem: any power of $A \in \mathbb{R}^{N\times N}$, power $k \geq N$

$\Rightarrow (A^T)^{k-1} = a_0 I^T + a_1 A^T + a_2 A^TA^T + \cdots a_{N-1}A^T{}^{N-1}$

assume $A_0 = A_1 = A_2 = \cdots = A_{K-1} = A$

& $k \gg N$

we first hv

$(A^T)^{k-1}C^T$

$= a_0 I C^T + a_1 A^T C^T + a_2 A^TA^TC^T + \cdots + a_{N-1}(A^T)^{N-1}C^T$

$\text{rank}[C_0^T A_0^T C_1^T\ A_0^T A_1^T C_2^T \cdots A_0^T \cdots A_{K-1}^T C_K^T]$

$\Rightarrow \text{rank}[C^T\ A^TC^T\ A^TA^TC^T \cdots (A^T)^K C^T]$

$= \text{rank}[C^T\ A^TC^T \cdots (A^T)^{N-1}C^T]$

- $$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{(N-1)} \end{bmatrix}$$

- $\text{rank}\ \mathcal{O} = N$ (observability)

$\therefore$ $Q_k \succ 0$, $R_k \succ 0$, $\text{rank}\ \mathcal{O} = N$ $\Rightarrow$ hv sol.

Bayesian Inference

compute full Bayesian Posterior

$P(x|v,y)$

- $x_k = A_{k-1}x_{k-1} + v_k + w_k$

$\xrightarrow{\text{lift}} x = A(v+w)$

where

$$A = \begin{bmatrix} 1 & & & & \\ A_0 & 1 & & & \\ A_1A_0 & A_1 & 1 & & \\ \vdots & \vdots & & \ddots & \\ A_{K-2}\cdots A_0 & A_{K-2}\cdots A_1 & A_{K-2}\cdots A_2 & \cdots & 1 \\ A_{K-1}\cdots A_0 & A_{K-1}\cdots A_1 & A_{K-1}\cdots A_2 & \cdots A_{K-1} & 1 \end{bmatrix}$$

- $\check{x} = E[x] = E[A(v+w)] = E[Av + Aw] = Av$

- $\check{P} = E[(x-E[x])(x-E[x])^T] = AQA^T$

$Q = E[ww^T] = \text{diag}(\check{P}_0, Q_1, \dots Q_K)$

- prior $\therefore \Rightarrow$

$$P(x|v) = \mathcal{N}(\check{x}, \check{P}) = \mathcal{N}(Av, AQA^T)$$

- $y_k = C_k x_k + n_k$

$\xrightarrow{\text{lift}} y = Cx + n$

where

$C = \text{diag}(C_0, C_1, \dots, C_K)$

- joint density of prior lifted state & measurements is now:

$$P(x, y|v) = \mathcal{N}\left(\begin{bmatrix}\check{x} \\ C\check{x}\end{bmatrix}, \begin{bmatrix}\check{P} & \check{P}C^T \\ C\check{P} & C\check{P}C^T + R\end{bmatrix}\right)$$

$$R = E[nn^T] = \text{diag}(R_0, R_1, \dots, R)$$

- $P(x, y|v) = \underbrace{P(x|y, v)}_{\text{we only care about this}} P(y|v)$

- $P(x, y) = \mathcal{N}\left(\begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}, \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}\right)$

- $P(x, y) = P(x|y)P(y)$

w/ schur complement

$$\begin{bmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix} = \begin{bmatrix} I & \Sigma_{xy}\Sigma_{yy}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} & 0 \\ 0 & \Sigma_{yy}\end{bmatrix}\begin{bmatrix} I & 0 \\ \Sigma_{yy}^{-1}\Sigma_{yx} & I\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -\Sigma_{yy}^{-1}\Sigma_{yx} & I \end{bmatrix}\begin{bmatrix} (\Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})^{-1} & 0 \\ 0 & \Sigma_{yy}^{-1}\end{bmatrix}\begin{bmatrix} I & -\Sigma_{xy}\Sigma_{yy}^{-1} \\ 0 & I\end{bmatrix}$$

- PDF of $P(x, y)$

recall:

$$P(x) = (2\pi)^{-\frac{k}{2}}\det(\Sigma)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

$$\Rightarrow P(x, y) = (2\pi)^{-\frac{k}{2}}\det(\Sigma)^{-\frac{1}{2}}\exp(Z)$$

$$Z = \left(\begin{bmatrix}x \\ y\end{bmatrix} - \begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}\right)^T\begin{pmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{pmatrix}^{-1}\left(\begin{bmatrix}x \\ y\end{bmatrix} - \begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}\right)$$

$$= \left(\begin{bmatrix}x \\ y\end{bmatrix} - \begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}\right)^T\begin{bmatrix} I & 0 \\ -\Sigma_{yy}^{-1}\Sigma_{yx} & I\end{bmatrix}\begin{bmatrix}(\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})^{-1} & 0 \\ 0 & \Sigma_{yy}^{-1}\end{bmatrix}\begin{bmatrix} I & -\Sigma_{xy}\Sigma_{yy}^{-1} \\ 0 & I\end{bmatrix} \times \left(\begin{bmatrix}x \\ y\end{bmatrix} - \begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}\right)$$

$$= \left([x-\mu_x-\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y)]^T(\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})^{-1} \times [x-\mu_x-\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y)]\right) + \left((y-\mu_y)^T\Sigma_{yy}^{-1}(y-\mu_y)\right)$$

sum of two quadratic terms

$P(x, y) = P(x|y)P(y)$

$P(x|y) = \mathcal{N}(\mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})$

$P(y) = \mathcal{N}(\mu_y, \Sigma_{yy})$

$$P(x, y|v) = P(x|y, v)P(y|v)$$

$$\therefore P(x|y, v)$$

recall $P(x, y) = \mathcal{N}(\mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})$

&

$$P(x, y|v) = \mathcal{N}\left(\begin{bmatrix}\check{x} \\ C\check{x}\end{bmatrix}, \begin{bmatrix}\check{P} & \check{P}C^T \\ C\check{P} & C\check{P}C^T+R\end{bmatrix}\right)$$

$$= \mathcal{N}\left(\check{x} + \check{P}C^T(C\check{P}C^T+R)^{-1}(y - C\check{x}),\ \check{P} - \check{P}C^T(C\check{P}C^T+R)^{-1}(C\check{P})\right)$$

thru Sherman-Morrison-Woodbury Identity

$$= \mathcal{N}\left((\check{P}^{-1} - C^TR^{-1}C)^{-1}(\check{P}^{-1}\check{x} + C^TR^{-1}y),\ (\check{P}^{-1} + C^TR^{-1}C)^{-1}\right)$$

i.e., $\hat{x} = (\check{P}^{-1} - C^TR^{-1}C)^{-1}(\check{P}^{-1}\check{x} + C^TR^{-1}y)$

$$\Rightarrow (\check{P}^{-1} - C^TR^{-1}C)\hat{x} = \check{P}^{-1}\check{x} + C^TR^{-1}y$$

$$\Rightarrow ((AQA^T)^{-1} - C^TR^{-1}C)\hat{x} = (AQA^T)^{-1}Av + C^TR^{-1}y$$

$$\Rightarrow (A^{-T}Q^{-1}A^{-1} - C^TR^{-1}C)\hat{x} = A^{-T}Q^{-1}A^{-1}Av + C^TR^{-1}y$$

$$\Rightarrow (A^{-T}Q^{-1}A^{-1} - C^TR^{-1}C)\hat{x} = A^{-T}Q^{-1}v + C^TR^{-1}y$$

$$A^{-1} = \begin{bmatrix} 1 & & & & \\ -A_0 & 1 & & & \\ & -A_1 & 1 & & \\ & & -A_2 & \ddots & \\ & & & & 1 \\ & & & -A_{K-1} & 1 \end{bmatrix}, \quad z = \begin{bmatrix} v \\ y \end{bmatrix}, \quad H = \begin{bmatrix} A^{-1} \\ C \end{bmatrix}, \quad W = \begin{bmatrix} Q \\ R \end{bmatrix}$$

$$\Rightarrow (H^T W^{-1} H)\hat{x} = H^T W^{-1} z$$

$A \to F$ $\quad y \to z$

$C \to H$ $\quad z \to y$

$V \to Bu$

$H \to \nabla$

Solving $(H^TW^{-1}H)\hat{x} = H^TW^{-1}z$

- Cholesky Decomposition

$H^TW^{-1}H = LL^T$ Cholesky Decomp.

- start decompose this
- $Ax = b$

$LL^Tx = b$

$\Rightarrow Ly = b$

- $L^Tx = y$
- solve x!

$\therefore (H^TW^{-1}H)\hat{x} = H^TW^{-1}z$

$\Rightarrow LL^T\hat{x} = H^TW^{-1}z$

$\Rightarrow Ld = H^TW^{-1}z$

$\Rightarrow L^T\hat{x} = d$

- let

$$L = \begin{bmatrix} L_0 & & & & & \\ L_{10} & L_1 & & & & \\ & L_{21} & L_2 & & & \\ & & \ddots & \ddots & & \\ & & & L_{K-1,K-2} & L_{K-1} & \\ & & & & L_{K,K-1} & L_K \end{bmatrix}$$

LL^T

$$L_0L_0^T = \underbrace{\check{P}_0^{-1} + C_0^TR_0^{-1}C_0}_{I_0} + A_0^TQ_1^{-1}A_0$$

$[A^{-1} + BD^{-1}C]^{-1} = A - AB(D + CAB)^{-1}CA$

$$L_{10}L_0^T = -Q_1^{-1}A_0$$

$$L_1L_1^T = -L_{10}L_{10}^T + Q_1^{-1} + C_1^TR_1^{-1}C_1 + A_1^TQ_2^{-1}A_1$$

$$L_{21}L_1^T = -Q_2^{-1}A_1$$

$\vdots$

$$L_{K-1}L_{K-1}^T = \underbrace{-L_{K-1,K-2}L_{K-1,K-2}^T + Q_{K-1}^{-1} + C_{K-1}^TR_{K-1}^{-1}C_{K-1}}_{I_{K-1}} + A_{K-1}^TQ_K^{-1}A_{K-1}$$

$$L_{K,K-1}L_{K-1}^T = -Q_K^{-1}A_{K-1}$$

$$L_KL_K^T = \underbrace{-L_{K,K-1}L_{K,K-1}^T + Q_K^{-1} + C_K^TR_K^{-1}C_K}_{I_K}$$

$I_0, I_1, \dots, I_{K-1}, I_K \Rightarrow$ Information Matrix

Basically solve L_0 → solve L_{10} → solve L_1 ⋮ $\quad O(N(K+1))$

forward

- $Ld = H^TW^{-1}z$ get $d = [d_0\ d_1 \dots d_K]^T$

$$L_0d_0 = \underbrace{\check{P}_0^{-1}\check{x}_0 + C_0^TR_0^{-1}y_0}_{q_0} - A_0^TQ_1^{-1}v_1$$

$$L_1d_1 = \underbrace{-L_{10}d_0 + Q_1^{-1}v_1 + C_1^TR_1^{-1}y_1}_{q_1} - A_1^TQ_2^{-1}v_2$$

$\vdots$

$$L_{K-1}d_{K-1} = \underbrace{-L_{K-1,K-2}d_{K-2} + Q_{K-1}^{-1}v_{K-1} + C_{K-1}^TR_{K-1}^{-1}y_{K-1}}_{q_{K-1}} - A_{K-1}^TQ_K^{-1}v_K$$

$$L_Kd_K = \underbrace{-L_{K,K-1}d_{K-1} + Q_K^{-1}v_K + C_K^TR_K^{-1}y_K}_{q_K}$$

Similarly, solve d_0 → solve d_1 → solve d_2 ⋮ $\quad O(N(K+1))$

backward

- $L^T\hat{x} = d$

$L_K^T\hat{x}_K = d_K$

$L_{K-1}^T\hat{x}_{K-1} = -L_{K,K-1}^T\hat{x}_K + d_{K-1}$

$\vdots$

$L_1^T\hat{x}_1 = -L_{21}^T\hat{x}_2 + d_1$

$L_0^T\hat{x}_0 = -L_{10}^T\hat{x}_1 + d_0$

Similarly: solve $\hat{x}_K$ → solve $\hat{x}_{K-1}$ → solve $\hat{x}_{K-2}$ ⋮ $\quad O(N(K+1))$

Forward

- $L_{K-1}L_{K-1}^T = I_{K-1} + A_{K-1}^TQ_K^{-1}A_{K-1}$

$L_{K-1}d_{K-1} = q_{K-1} - A_{K-1}^TQ_K^{-1}v_K$

$L_{K,K-1}L_{K-1}^T = -Q_K^{-1}A_{K-1}$

$I_K = -L_{K,K-1}L_{K,K-1}^T + Q_K^{-1} + C_K^TR_K^{-1}C_K$

$q_K = -L_{K,K-1}d_{K-1} + Q_K^{-1}v_K + C_K^TR_K^{-1}y_K$

Backward

- $L_{K-1}^T\hat{x}_{K-1} = -L_{K,K-1}^T\hat{x}_K + d_{K-1}$

Kalman filter

△ solving $(H^TW^{-1}H)\hat{x} = H^TW^{-1}z$

• Rauch-Tung-Striebel

from above:

$L_{k-1}L_{k-1}^T = I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1}$

$L_{k,k-1}L_{k-1}^T = -Q_k^{-1}A_{k-1}$

$I_k = -L_{k,k-1}L_{k,k-1}^T + Q_k^{-1} + C_k^TR_k^{-1}C_k$

∴ $L_{k,k-1} = -Q_k^{-1}A_{k-1}L_{k-1}^{-T}$

$L_{k,k-1}L_{k,k-1}^T = Q_k^{-1}A_{k-1}L_{k-1}^{-T}L_{k-1}^{-1}A_{k-1}^TQ_k^{-T}$

$= Q_k^{-1}A_{k-1}(L_{k-1}L_{k-1}^T)^{-1}A_{k-1}^TQ_k^{-T}$

∴ $I_k = -L_{k,k-1}L_{k,k-1}^T + Q_k^{-1} + C_k^TR_k^{-1}C_k$

$= Q_k^{-1} - Q_k^{-1}A_{k-1}(I_{k-1}+A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}A_{k-1}^TQ_k^{-T} + C_k^TR_k^{-1}C_k$

$= (A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1} + C_k^TR_k^{-1}C_k$

from Sherman-Morrison-Woodbury

$I_k = (A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1} + C_k^TR_k^{-1}C_k$

$\hat{P}_{k,f} = I_k^{-1}$

$\check{P}_{k,f} = A_{k-1}\hat{I}_{k-1}^{-1}A_{k-1}^TQ_k$

$\hat{P}_{k,f}^{-1} = \check{P}_{k,f}^{-1} + C_k^TR_k^{-1}C_k$

△ inducing KF

let us define

$K_k = \hat{P}_{k,f}C_k^TR_k^{-1}$

$= (\check{P}_{k,f}^{-1} + C_k^TR_k^{-1}C_k)^{-1}C_k^TR_k^{-1}$ SMW

$= \check{P}_{k,f}C_k^T(C_k\check{P}_{k,f}C_k^T + R_k)^{-1}$

which is the Kalman gain that we are familiar of

- $\check{P}_{k,f}^{-1} = \hat{P}_{k,f}^{-1} - C_k^TR_k^{-1}C_k$

$= \hat{P}_{k,f}^{-1}[1 - \hat{P}_{k,f}C_k^TR_k^{-1}C_k]$

$= \hat{P}_{k,f}^{-1}(1 - K_kC_k)$

⇒ $\hat{P}_{k,f} = (1 - K_kC_k)\check{P}_{k,f}$

- now, from above

$$\begin{cases} L_{k,k-1}L_{k-1}^T = -Q_k^{-1}A_{k-1} \\ L_{k-1}d_{k-1} = q_{k-1} - A_{k-1}^TQ_k^{-1}v_k \\ L_{k-1}L_{k-1}^T = I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1} \end{cases}$$

$L_{k,k-1} = (-Q_k^{-1}A_{k-1})L_{k-1}^{-T}$

$d_{k-1} = L_{k-1}^{-1}(q_{k-1} - A_{k-1}^TQ_k^{-1}v_k)$

⇒ $L_{k,k-1}d_{k-1} = (-Q_k^{-1}A_{k-1})L_{k-1}^{-T}L_{k-1}^{-1} \times (q_{k-1} - A_{k-1}^TQ_k^{-1}v_k)$

⇒ $L_{k,k-1}d_{k-1} = (-Q_k^{-1}A_{k-1})(L_{k-1}L_{k-1}^T)^{-1}(q_{k-1} - A_{k-1}^TQ_k^{-1}v_k)$ (1)

∵ $q_k = -L_{k,k-1}d_{k-1} + Q_k^{-1}v_k + C_k^TR_k^{-1}y_k$ (2)

combining (1) & (2)

$q_k = Q_k^{-1}A_{k-1}(I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}q_{k-1}$
$+ (Q_k^{-1} - Q_k^{-1}A_{k-1}(I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}A_{k-1}^TQ_k^{-1})v_k$
$+ C_k^TR_k^{-1}y_k$ thru SMW

$= (A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1}A_{k-1}I_{k-1}^{-1}q_{k-1}$
$+ (A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1}v_k$
$+ C_k^TR_k^{-1}y_k$

$= (A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1}(A_{k-1}I_{k-1}^{-1}q_{k-1} + v_k)$
$+ C_k^TR_k^{-1}y_k$

$= (\check{P}_{k,f})^{-1}(A_{k-1}\hat{P}_{k-1,f}\hat{P}_{k-1,f}^{-1}\hat{x}_{k-1,f} + v_k)$
$+ C_k^TR_k^{-1}y_k$ let $q_k = \hat{P}_{k,f}^{-1}\hat{x}_{k,f}$

⇒ $\hat{P}_{k,f}^{-1}\hat{x}_{k,f} = \check{P}_{k,f}^{-1}(A_{k-1}\hat{x}_{k-1,f} + v_k)$
$+ C_k^TR_k^{-1}y_k$

⇒ $\hat{P}_{k,f}^{-1}\hat{x}_{k,f} = \check{P}_{k,f}^{-1}\check{x}_{k,f} + C_k^TR_k^{-1}y_k$

⇒ $\hat{x}_{k,f} = \hat{P}_{k,f}\check{P}_{k,f}^{-1}\check{x}_{k,f} + \hat{P}_{k,f}C_k^TR_k^{-1}y_k$

$= (1 - K_kC_k)\check{x}_{k,f} + K_ky_k$

⇒ $\hat{x}_{k,f} = \check{x}_{k,f} + K_k(y_k - C_k\check{x}_{k,f})$

end of "forward"

• backward

from above

$L_{k-1}^T\hat{x}_{k-1} = -L_{k,k-1}^T\hat{x}_k + d_{k-1}$

⇒ $\hat{x}_{k-1} = (L_{k-1}^T)^{-1}(-L_{k,k-1}^T\hat{x}_k + d_{k-1})$

$= L_{k-1}^{-T}L_{k-1}^{-1}L_{k-1}(-L_{k,k-1}^T\hat{x}_k + d_{k-1})$

$= (L_{k-1}L_{k-1}^T)^{-1}L_{k-1}(-L_{k,k-1}^T\hat{x}_k + d_{k-1})$

⇒ from

$L_{k-1}L_{k-1}^T = I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1}$

$L_{k-1}d_{k-1} = q_{k-1} - A_{k-1}^TQ_k^{-1}v_k$

$L_{k,k-1}L_{k-1}^T = -Q_k^{-1}A_{k-1}$

$= (L_{k-1}L_{k-1}^T)^{-1}(-L_{k-1}L_{k,k-1}^T\hat{x}_k$
$+ L_{k-1}d_{k-1})$

$= (I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}$
$\times (-(L_{k,k-1}L_{k-1}^T)^T\hat{x}_k$
$+ q_{k-1} - A_{k-1}^TQ_k^{-1}v_k)$

$= (I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}$
$\times (A_{k-1}^TQ_k^{-1}\hat{x}_k$
$+ q_{k-1} - A_{k-1}^TQ_k^{-1}v_k)$

$= (I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}$
$\times (A_{k-1}^TQ_k^{-1}(\hat{x}_k - v_k) + q_{k-1}$

$= (I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}A_{k-1}^TQ_k^{-1}(\hat{x}_k - v_k)$
$+ (I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}q_{k-1}$

$(I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}A_{k-1}^TQ_k^{-1}(\hat{x}_k - v_k)$
$+ (I_{k-1} + A_{k-1}^TQ_k^{-1}A_{k-1})^{-1}q_{k-1}$ thru SMW

$= I_{k-1}^{-1}A_{k-1}^T(A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1}(\hat{x}_k - v_k)$
$+ (I_{k-1}^{-1} - I_{k-1}^{-1}A_{k-1}^T(A_{k-1}I_{k-1}^{-1}A_{k-1}^T + Q_k)^{-1}A_{k-1}I_{k-1}^{-1})q_{k-1}$

$= \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}(\hat{x}_k - v_k)$
$+ (\hat{P}_{k-1,f} - \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}A_{k-1}\hat{P}_{k-1,f})\hat{P}_{k-1,f}^{-1}\hat{x}_{k-1,f}$

$= \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}(\hat{x}_k - v_k)$
$+ \hat{x}_{k-1,f} - \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}A_{k-1}\cancel{\hat{P}_{k-1,f}\hat{P}_{k-1,f}^{-1}}\hat{x}_{k-1,f}$

$= \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}(\hat{x}_k - v_k)$
$+ \hat{x}_{k-1,f} - \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}\check{x}_{k,f}$

$= \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}(\hat{x}_k - \check{x}_{k,f} - v_k) + \hat{x}_{k-1,f}$

$= \hat{P}_{k-1,f}A_{k-1}^T\check{P}_{k,f}^{-1}(\hat{x}_k - \check{x}_{k,f}) + \hat{x}_{k-1,f}$

Non-linear Estimations

Recall

J, cost

$$J_{v,k}(x) = \begin{cases} \frac{1}{2}(x_0 - \check{x}_0)^T\check{P}_0^{-1}(x_0 - \check{x}_0) \\ \frac{1}{2}(x_k - A_{k-1}x_{k-1} - v_k)^TQ_k^{-1}(x_k - A_{k-1}x_{k-1} - v_k) \end{cases}$$

$J_{y,k}(x) = \frac{1}{2}(y_k - C_kx_k)^TR_k^{-1}(y_k - C_kx_k)$

Mahalanobis distance

$$J(x) = \sum_{k=0}^{K}(J_{v,k}(x) + J_{y,k}(x))$$

∴ $x = \arg\min_x J(x)$

• here the system, i.e.,
- the prediction model
- the measurement model
is no longer linear

△
$$\begin{cases} e_{v,k}(x) = \begin{cases} \check{x}_0 - x_0 & , k=0 \\ f(x_{k-1}, v_k, 0) - x_k & , k=1\sim K \end{cases} \\ e_{y,k}(x) = y_k - g(x_k, 0), \quad k=0\sim K \end{cases}$$

$$\begin{cases} J_{v,k}(x) = \frac{1}{2}e_{v,k}(x)^TW_{v,k}^{-1}e_{v,k}(x) \\ J_{y,k}(x) = \frac{1}{2}e_{y,k}(x)^TW_{y,k}^{-1}e_{y,k}(x) \end{cases}$$

⇒ $J(x) = \sum_{k=0}^{K}(J_{v,k}(x) + J_{y,k}(x))$

⇒ • let $e(x) = \begin{bmatrix} e_v(x) \\ e_y(x) \end{bmatrix}$

$e_v(x) = \begin{bmatrix} e_{v,0}(x) \\ \vdots \\ e_{v,K}(x) \end{bmatrix}$

$e_y(x) = \begin{bmatrix} e_{y,0}(x) \\ \vdots \\ e_{y,K}(x) \end{bmatrix}$

• let $W = \mathrm{diag}(W_v, W_y)$

$W_v = \mathrm{diag}(W_{v,0}, \ldots W_{v,K})$

$W_y = \mathrm{diag}(W_{y,0}, \ldots W_{y,K})$

• $J(x) = \frac{1}{2}e(x)^TW^{-1}e(x)$

⇒ let $u(x) = Le(x)$

$L^TL = W^{-1}$

$J(x) = \frac{1}{2}e(x)^TL^TLe(x)$

$= \frac{1}{2}[Le(x)]^TLe(x)$

$= \frac{1}{2}u(x)^Tu(x)$

⇒ $\hat{x} = \arg\min_x J(x)$

Solving $\hat{x} = \arg\min_x J(x)$

• Newton's Method

@ update step

$J(x_{op} + \delta x)$

$\approx J(x_{op}) + \left(\frac{\partial J(x)}{\partial x}\Big|_{x_{op}}\right)\delta x + \frac{1}{2}\delta x^T\left(\frac{\partial^2J(x)}{\partial x\partial x^T}\Big|_{x_{op}}\right)\delta x$

$\frac{\partial J(x_{op} + \delta x)}{\partial \delta x}$

$= \left(\frac{\partial J(x)}{\partial x}\Big|_{x_{op}}\right) + \delta x^{*T}\left(\frac{\partial^2J(x)}{\partial x\partial x^T}\Big|_{x_{op}}\right) = 0$

⇒ $\left(\frac{\partial^2J(x)}{\partial x\partial x^T}\Big|_{x_{op}}\right)\underbrace{\delta x^*}_{\text{update step}} = -\left(\frac{\partial J(x)}{\partial x}\Big|_{x_{op}}\right)^T$

then

$x_{op} \leftarrow x_{op} + \delta x^*$

• Gauss-Newton Method

from above

- $J(x) = \frac{1}{2}u(x)^Tu(x)$
- Jacobian

$\frac{\partial J(x)}{\partial x}\Big|_{x_{op}} = u(x_{op})^T\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)$

- Hessian

$\frac{\partial^2J(x)}{\partial x\partial x^T}\Big|_{x_{op}}$

$= \left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)^T\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right) + \underbrace{\sum_{i=1}^{M}u_i(x_{op})\left(\frac{\partial^2u_i(x)}{\partial x\partial x^T}\Big|_{x_{op}}\right)}_{\text{neglect this}}$

$\approx \left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)^T\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)$

∴ $\left(\frac{\partial^2J(x)}{\partial x\partial x^T}\Big|_{x_{op}}\right)\delta x^* = -\left(\frac{\partial J(x)}{\partial x}\Big|_{x_{op}}\right)^T$

⇒ $\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)^T\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)\delta x^* = -\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)^Tu(x_{op})$

△ GN method

recall:

• $\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)^T\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)\delta x^* = -\left(\frac{\partial u(x)}{\partial x}\Big|_{x_{op}}\right)^Tu(x_{op})$

• $J(x) = \frac{1}{2}e(x)^TW^{-1}e(x)$

$= \frac{1}{2}e(x)^TL^TLe(x)$

$= \frac{1}{2}[Le(x)]^TLe(x)$

$= \frac{1}{2}u(x)^Tu(x)$

• $$\begin{cases} e_{v,k}(x) = \begin{cases} \check{x}_0 - x_0 & , k=0 \\ f(x_{k-1}, v_k, 0) - x_k & , k=1\sim K \end{cases} \\ e_{y,k}(x) = y_k - g(x_k, 0), \quad k=0\sim K \end{cases}$$

NOW:

• for any function $f(x)$, perturbed by δx @ a
we can write Taylor expansion: (2nd order)

$f(a + \delta x)$

$= f(a) + \frac{\partial f(x)}{\partial x}\Big|_{x=a}\delta x + \frac{1}{2}\delta x^T\frac{\partial^2f(x)}{\partial x\partial x^T}\Big|_{x=a}\delta x$

∴ w/ 1st order

$$e_{v,k}(x_{op} + \delta x) \approx \begin{cases} e_{v,0}(x_{op,0}) + \frac{\partial e_{v,0}(x)}{\partial x}\Big|_{x_{op,0}}\delta x, & k=0 \\ e_{v,k}(x_{op,k}) + \frac{\partial e_{v,k}(x)}{\partial x}\Big|_{x_{op,k}}\delta x, & k\neq 0 \end{cases}$$

$$\Rightarrow \begin{cases} ① \ \check{x}_0 - x_{op,0} - 1\cdot\delta x & , k=0 \\ ② \ f(x_{op,k-1}, v_k, 0) - x_{op,k} + F_{k-1}\delta x_{k-1} - \delta x_k, & k\neq 0 \end{cases}$$

$F_{k-1} = \frac{\partial f(x_{k-1}, v_k, w_k)}{\partial x_{k-1}}\Big|_{x_{op,k-1}, v_k, 0}$

$e_{y,k}(x_{op} + \delta x) \approx e_{y,k}(x_{op}) + \frac{\partial e_{y,k}(x)}{\partial x}\Big|_{x_{op,k}}$

⇒ ③ $y_k - g(x_{op,k}, 0) - G_k\delta x_k$

$G_k = \frac{\partial g(x_k, n_k)}{\partial x_k}\Big|_{x_{op,k}, 0}$

THEN

$J(x) = \frac{1}{2}e(x)^TW^{-1}e(x)$

- stack all $e(x)$'s

$e(x_{op}) = \begin{bmatrix} e_{v,0}(x_{op}) \\ e_{v,1}(x_{op}) \\ \vdots \\ e_{v,K}(x_{op}) \\ e_{y,0}(x_{op}) \\ e_{y,1}(x_{op}) \\ \vdots \\ e_{y,K}(x_{op}) \end{bmatrix}$

- WHILE in iterative method, variable is δx

$\delta x = \begin{bmatrix} \delta x_0 \\ \delta x_1 \\ \vdots \\ \delta x_K \end{bmatrix}$

- ALSO, recall GN:

$J(x)^TJ(x)\delta x = -J(x)^Tf(x)$

HERE ∴ $J(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}$

"$f(x)$" $= e(x_{op}) = \begin{bmatrix} e_{v,0}(x_{op}) \\ e_{v,1}(x_{op}) \\ \vdots \\ e_{v,K}(x_{op}) \\ e_{y,0}(x_{op}) \\ e_{y,1}(x_{op}) \\ \vdots \\ e_{y,K}(x_{op}) \end{bmatrix}$

"x" $= \delta x = \begin{bmatrix} \delta x_0 \\ \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_K \end{bmatrix}$

"$J(x)$"

$= J$

$= \begin{bmatrix} \frac{\partial e_{v,0}(x_{op})}{\partial \delta x_0} & \frac{\partial e_{v,0}(x_{op})}{\partial \delta x_1} & \cdots \\ \frac{\partial e_{v,1}(x_{op})}{\partial \delta x_0} & \frac{\partial e_{v,1}(x_{op})}{\partial \delta x_1} & \cdots \\ \vdots & \frac{\partial e_{v,2}(x_{op})}{\partial \delta x_2} & \\ \cdots & & \ddots \\ \frac{\partial e_{y,0}(x_{op})}{\partial \delta x_0} & \frac{\partial e_{y,0}(x_{op})}{\partial \delta x_1} & \cdots \\ \frac{\partial e_{y,1}(x_{op})}{\partial \delta x_0} & \frac{\partial e_{y,1}(x_{op})}{\partial \delta x_1} & \cdots \\ \vdots & \vdots & \ddots \\ \cdots & \cdots & \end{bmatrix}$

$= \begin{bmatrix} F_0 & 1 & & \\ & F_1 & \ddots & \\ & & F_{K-1} & 1 \\ -G_0 & & & \\ & -G_1 & & \\ & & \ddots & \end{bmatrix}$

Marginalization - Sliding Window

recall the linear system

$H\delta x = b$

ALSO recall Schur complement

$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

$S = C - B^TA^{-1}C$

↳ Schur complement of A in X

• The core idea of marginalization in SWF:

• $H\delta x = b$ — we want this out (δx_a)

⇒ $\begin{bmatrix} \Lambda_a & \Lambda_b \\ \Lambda_b^T & \Lambda_c \end{bmatrix}\begin{bmatrix} \delta x_a \\ \delta x_b \end{bmatrix} = \begin{bmatrix} g_a \\ g_b \end{bmatrix}$

⇒ $\begin{bmatrix} \Lambda_a & \Lambda_b \\ \Lambda_b^T - \Lambda_b^T\Lambda_a^{-1}\Lambda_a & \Lambda_c - \Lambda_b^T\Lambda_a^{-1}\Lambda_b \end{bmatrix}\begin{bmatrix} \delta x_a \\ \delta x_b \end{bmatrix} = \begin{bmatrix} g_a \\ g_b - \Lambda_b^T\Lambda_a^{-1}g_a \end{bmatrix}$

⇒ $\begin{bmatrix} \Lambda_a & \Lambda_b \\ 0 & \Lambda_c - \Lambda_b^T\Lambda_a^{-1}\Lambda_b \end{bmatrix}\begin{bmatrix} \delta x_a \\ \delta x_b \end{bmatrix} = \begin{bmatrix} g_a \\ g_b - \Lambda_b^T\Lambda_a^{-1}g_a \end{bmatrix}$

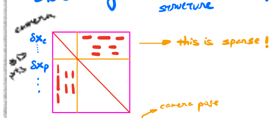
⇒ $(\Lambda_c - \Lambda_b^T\Lambda_a^{-1}\Lambda_b)\delta x_b = g_b - \Lambda_b^T\Lambda_a^{-1}g_a$

↳ we get marginalized state!

ALSO! sparse ⟶ dense

• in pose graph optimization

• Generally H matrix has following structure



⇒ $\begin{bmatrix} B & E \\ E^T & C \end{bmatrix}\begin{bmatrix} \Delta x_c \\ \Delta x_p \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$ (camera pose; 3D pts)

⇒ $\begin{bmatrix} B - EC^{-1}E^T & E - EC^{-1}C \\ E^T & C \end{bmatrix}\begin{bmatrix} \Delta x_c \\ \Delta x_p \end{bmatrix} = \begin{bmatrix} v - EC^{-1}w \\ w \end{bmatrix}$

⇒ $\begin{bmatrix} B - EC^{-1}E^T & 0 \\ E^T & C \end{bmatrix}\begin{bmatrix} \Delta x_c \\ \Delta x_p \end{bmatrix} = \begin{bmatrix} v - EC^{-1}w \\ w \end{bmatrix}$

$$\begin{cases} (B - EC^{-1}E^T)\Delta x_c = v - EC^{-1}w \\ E^T\Delta x_c + C\Delta x_p = w \end{cases}$$

C^{-1} (diagnol) ⇒ easy to solve
then solve Δx_p!

$(J^TW^{-1}J)\delta x = -J^TW^{-1}e(x_{op})$

remember $u(x) = Le(x)$
previously, I dropped "L"

@ iteration step i

solve δx

THEN

$x_{op} \leftarrow x_{op} + \delta x$

★ solving the linear system @ i, has similar formulation w/ LGE

★ NLGE ⟷ LGE

- iterative linearization
- solve the linear system locally @ i

- solve the linear system globally

★ $\delta x^k = \arg\min_{\delta x} \sum \|e - J\delta x\|_\Lambda^2$