- Optimization problem

$$\min_{} f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0, \; i=1,\dots,m; \quad h_i(x)=0, \; i=1,\dots,p$$

- duality:
  - gives us lower bound
  - sometimes it's easier to solve (good properties for solver!)

- dual problem
  - $L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p}\nu_i h_i(x)$
  - $g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu) = \inf_{x\in D}\left\{f_0(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p}\nu_i h_i(x)\right\}$
  - maximize $g(\lambda,\nu)$ s.t. $\lambda \succeq 0$

- QCQP (quadratic constrained quadratic programming)

▲ minimize$_x$ x^TQx
s.t. $x^TA_kx = b_k \quad \forall k \in \{1,\dots,k\}$

non-convex QCQP

Lagrange multiplier

▲ $L(x,\lambda) = x^TQx + \sum_{k=1}^{K}\lambda_k(b_k - x^TA_kx)$

$= \sum_k \lambda_k b_k + x^T\left(Q - \sum_{k=1}^{K}\lambda_k A_k\right)x$

$= \underbrace{b^T\lambda}_{\begin{bmatrix} b_1 \\ \vdots \\ b_k\end{bmatrix}^T \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_k\end{bmatrix}} + x^T \underbrace{H(\lambda)}_{Q-\sum\lambda_k A_k} x$

▲ $g(\lambda) = \inf_x L(x,\lambda) = \begin{cases} b^T\lambda, & \text{if } H(\lambda)\succeq 0 \\ -\infty, & \text{o.w.}\end{cases} \rightarrow \frac{\partial L}{\partial x} = 0$

▲ primal problem

minimize$_x$ x^TQx
s.t. $x^TA_kx = b_k \quad \forall k \in \{1,\dots,K\}$

⇕

$\theta_P(w) = \max_\lambda \mathcal{L}(x,\lambda)$

$= \max_\lambda \begin{cases} x^TQx & x\in D \\ \infty & \text{o.w.}\end{cases}$

$\left(\text{minimize}_x \; \max_\lambda \left(b^T\lambda + x^T\underbrace{H(\lambda)x}_{0}\right) \quad (1)\right)$

▲ dual problem

$\left(\begin{array}{l} \text{maximize } b^T\lambda \\ \text{s.t. } H(\lambda)\succeq 0 \\ \quad\;\; H(\lambda)x = 0 \end{array} \quad (2)\right)$

∴ from (1) & (2)

when $H(\lambda)x = 0$

$p^* = d^* = b^T\lambda$

strong duality holds → Slater's condition: $\exists x \in \text{relint } D$ s.t. $f_i(x) < 0$, $Ax = b$

empirically: robotics application satisfy Slater's condition

This implies:
a. strong duality (if $H(\lambda)\succeq 0$)
b. $\hat{x}$ is optimal (global) when $H\hat{x} = 0$
c. $\hat{x}$ is not optimal then what do we do?

▲ from above — certification problem

find H, λ
s.t. $H = Q - \sum_{k=1}^{K}\lambda_k A_k$
$H \succeq 0$
$H\hat{x} = 0$

strong duality holds!

↓

we have global optimal!

▲ SDP relaxation (which can allow us to solve it faster & solve it further!)
dual of dual problem
(omitted today)

minimize$_x$ x^TQx
s.t. $xAx = b$

- recall dual problem
maximize $b^T\lambda$
s.t. $H(\lambda)\succeq 0$

- Langragian:
$L'(\lambda, X) = b^T\lambda + \mathrm{tr}(XH(\lambda))$, $X \succeq 0$ matrix of Langragian multiplier

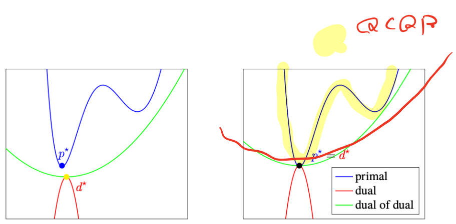
note: $\langle X, Y\rangle = \mathrm{tr}(X^TY)$

- dual problem:
  - $L'(\lambda, X) = b^T\lambda + \mathrm{tr}\left(X\left(Q - \sum_{k=1}^{K}\lambda_k A_k\right)\right)$

    $= \mathrm{tr}(QX) + [b_1 - \mathrm{tr}(A_1X) \quad b_k - \mathrm{tr}(A_kX)]\lambda$
  - $g(X) = \sup_\lambda L'(\lambda, X) = \begin{cases}\mathrm{tr}(QX) & \text{if } \mathrm{tr}(A_kX) = b_k \\ \infty & \text{o.w.}\end{cases}$

⇒ $\boxed{\begin{array}{l}\text{minimize}_X \; \mathrm{tr}(QX) \\ \quad \mathrm{tr}(A_kX) = b_k \\ \quad X \succeq 0\end{array}}$ Semidefinite Programming!

$xx^T = X$



(a) Weak duality. (b) Strong duality.