from Dr. Can

Kalman Filter 1. recursive algorithm

(an optimal recursive data processing algorithm)

an example when there are k measurements: intuitively, we take average

$\hat{x}_k = \frac{1}{k}(z_1+z_2+\cdots+z_k)$

$= \frac{1}{k}(z_1+z_2+\cdots+z_{k-1}) + \frac{1}{k}z_k$

$= \frac{1}{k}\cdot\frac{k-1}{k-1}(z_1+z_2+\cdots+z_{k-1}) + \frac{1}{k}z_k$

$= \frac{k-1}{k}\hat{x}_{k-1} + \frac{1}{k}z_k$

$= \hat{x}_{k-1} - \frac{1}{k}\hat{x}_{k-1} + \frac{1}{k}z_k$

$\Rightarrow \hat{x}_k = \hat{x}_{k-1} + \frac{1}{k}(z_k - \hat{x}_{k-1})$

$k\uparrow,\ \frac{1}{k}\to 0$ $\hat{x}_k \to \hat{x}_{k-1}$ (measurement less important)

$k\downarrow\ \frac{1}{k}\uparrow$ z_k more important

△ let $\frac{1}{k} := K_k$ K_k

$\therefore \hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$

△ include error into consideration

e_{est} e_{mea}

$K_k = \frac{e_{est\,k-1}}{e_{est\,k-1} + e_{mea\,k}}$

@k ① if $e_{est\,k-1} \gg e_{mea\,k}$:

$\to K_k \to 1$

$\to \hat{x}_k = \hat{x}_{k-1} + z_k - \hat{x}_{k-1} = z_k$

② if $e_{est\,k-1} \ll e_{mea\,k}$:

$\to K_k \to 0$

$\to \hat{x}_k = \hat{x}_{k-1}$

△ KF algorithm

① calculate $K_k = \frac{e_{est\,k-1}}{e_{est\,k-1} + e_{mea\,k}}$

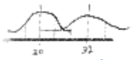
② calculate $\hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$

③ update $e_{est\,k} = (1-K_k)\,e_{est\,k-1}$

2. Data fusion / Covariance Matrix / State Space / Observation

△ Data Fusion

- e.g. $z_1 = 30\ g$ $\sigma_1 = 2\ g$; $z_2 = 32\ g$ $\sigma_2 = 4\ g$



• what is $\hat{z}$?

$\hat{z} = z_1 + K(z_2 - z_1)$

• then what is K? optimal K occurs @ $\sigma_{\hat{z}}$ has min.

$\sigma_{\hat{z}}^2 = Var(z_1 + K(z_2 - z_1))$

$= Var(z_1 + Kz_2 - Kz_1)$

$= Var((1-K)z_1 + Kz_2)$

$= Var((1-K)z_1) + Var(Kz_2)$

$= (1-K)^2 Var(z_1) + K^2 Var(z_2)$

$= (1-K)^2\sigma_1^2 + K^2\sigma_2^2$

$\Rightarrow$ min. value @ $\frac{d}{dK}\sigma_{\hat{z}}^2 = 0$

$\Rightarrow \frac{d}{dK}\sigma_{\hat{z}}^2 = -2(1-K)\sigma_1^2 + 2K\sigma_2^2 = 0$

$= -\sigma_1^2 + K\sigma_1^2 + K\sigma_2^2 = 0$

$\Rightarrow K(\sigma_1^2+\sigma_2^2) = \sigma_1^2$

$\Rightarrow K = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$

△ Covariance Matrix

	x (height)	y (weight)	z (age)
P1	179	74	33
P2	187	80	31
P3	175	71	28
avg	180.3	75	30.7

$\sigma_x^2 = \frac{1}{3}[(179-180.3)^2 + (187-180.3)^2 + (175-180.3)^2] = 24.89$

$\sigma_x\sigma_y = \frac{1}{3}[(179-180.3)(74-75) + (187-180.3)(80-75) + (175-180.3)(71-75)] = 18.7$

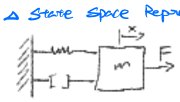
covariance matrix (contd)

• $P = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y & \sigma_x\sigma_z \\ \sigma_y\sigma_x & \sigma_y^2 & \sigma_y\sigma_z \\ \sigma_z\sigma_x & \sigma_z\sigma_y & \sigma_z^2 \end{bmatrix}$

• $a = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$

$P = \frac{1}{3}a^Ta$

△ State Space Representation



• $m\ddot{x} + B\dot{x} + kx = F = u$

dynamics

$x_1 = x$

$x_2 = \dot{x}$

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{dynamics w/o input}}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = \frac{1}{m}(F - B\dot{x} - kx)$

measurements $= \frac{1}{m}(F - Bx_2 - kx_1)$

• $z_1 = x = x_1$

$z_2 = \dot{x} = x_2$

• $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}u$

$\dot{x}_t = A\ x_t + B\ u_t$

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$z_t = H\ x_t$ → to discrete

$x_k = Ax_{k-1} + Bu_k + w_{k-1}$

$z_k = Hx_k + v_k$

$\therefore$ How to get $\hat{x}_k$

KF math

$x_k = Ax_{k-1} + Bu_k + w_k$

$z_k = Hx_k + v_k$

△ $P(w) \sim (0, Q)$

$Q = E[ww^T]$

• $E\left[\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\begin{bmatrix} w_1 & w_2 \end{bmatrix}\right]$ $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$= \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$

• $VAR = E[x^2] - E^2[x]$ $= 0$ for w_1 w_2

$\therefore E[a^2] = Var = \sigma^2$

$\therefore \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$

$= \begin{bmatrix} \sigma_{w_1}^2 & \sigma_{w_1}\sigma_{w_2} \\ \sigma_{w_2}\sigma_{w_1} & \sigma_{w_2}^2 \end{bmatrix}$

△ $P(v) \sim (0, R)$

△ Priori

$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$ ① calculated

$z_k = Hx_k \to \hat{x}_{k\,mea} = H^-z_k$ measured

△ Posteriori

$\hat{x}_k = \hat{x}_k^- + G(H^-z_k - \hat{x}_k^-)$

$G = K_kH$

$\Rightarrow \hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

△ infer K_k

get K_k s.t. $\hat{x}_k \to x_k$

• let $e_k = x_k - \hat{x}_k$

$P(e_k) \sim (0, P)$

$P = E[ee^T] = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} \\ \sigma_{e_2}\sigma_{e_1} & \sigma_{e_2}^2 \end{bmatrix}$

• objective: minimize $tr(P) = \sigma_{e_1}^2 + \sigma_{e_2}^2$

• $P = E[ee^T]$

$= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$

$x_k - \hat{x}_k = x_k - (\hat{x}_k^- + K_k(z_k - H\hat{x}_k^-))$

$= x_k - \hat{x}_k^- - K_kz_k + K_kH\hat{x}_k^-$

$= x_k - \hat{x}_k^- - K_kHx_k - K_kv_k + K_kH\hat{x}_k^-$

$= (x_k - \hat{x}_k^-) - K_kH(x_k - x_k^-) - K_kv_k$

$= (I - K_kH)(x_k - \hat{x}_k^-) - K_kv_k$

$= E\left[[(I-K_kH)e_k^- - K_kv_k][(I-K_kH)e_k^- - K_kv_k]^T\right]$

$= E[(I-K_kH)e_k^-e_k^{-T}(I-K_kH)^T$

$- \cancel{(I-K_kH)e_k^-}v_kv_k^T = 0$

$- K_kv_k\cancel{e_k^{-T}(I-K_kH)^T} = 0$

$+ K_kv_kv_k^TK_k^T]$

$= (I-K_kH)E(e_k^-e_k^{-T})(I-K_kH)^T$

$+ K_kE[v_kv_k^T]K_k^T$

$= (I-K_kH)P_k^-(I-K_kH)^T + K_kE[v_kv_k^T]K_k^T$

$= (P_k^- - K_kHP_k^-)(I-K_kH)^T + K_kRK_k^T$

$\Rightarrow P_k = P_k^- - K_kHP_k^- - P_k^-H^TK_k^T + K_kHP_k^-H^TK_k^T + K_kRK_k^T$

$\Rightarrow tr(P_k) = tr(P_k^-) - 2tr(K_kHP_k^-) + tr(K_kHP_k^-H^TK_k^T)$
$+ tr(K_kRK_k^T)$

$\frac{d\,tr(AB)}{dA} = B^T$

$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & \vdots \\ \vdots & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$

$\therefore tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$

$\therefore \frac{d\,tr(AB)}{dA} = \begin{bmatrix} \frac{\partial tr(AB)}{\partial a_{11}} & \frac{\partial tr(AB)}{\partial a_{12}} \\ \frac{\partial tr(AB)}{\partial a_{21}} & \frac{\partial tr(AB)}{\partial a_{22}} \end{bmatrix}$

$= B^T$

- $\frac{d\,tr(P_k)}{dK_k} = 0 - 2(HP_k^-)^T + 2K_kHP_k^-H^T + 2K_kR = 0$

$\therefore K_k = P_k^-H^T(HP_k^-H^T + R)^{-1}$

△ Priori/Posteriori Error Covariance Matrix

• recall

$x_k = Ax_{k-1} + Bu_k + w_k$ $w \sim P(0, Q)$

$z_k = Hx_k + v_k$ $v \sim P(0, R)$

• Priori

$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

Posteriori

$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

Kalman gain

$K_k = P_k^-H^T(HP_k^-H^T + R)^{-1}$

• $P_k^- := E[e_k^-e_k^{-T}]$

$\Rightarrow e_k^- = x_k - \hat{x}_k^-$

$= Ax_{k-1} + \cancel{Bu_k} + w_k - A\hat{x}_{k-1} - \cancel{Bu_k}$

$= A(x_{k-1} - \hat{x}_{k-1}) + w_k$

$= Ae_{k-1} + w_k$

$\therefore P_k^- = E[(Ae_{k-1} + w_k)(Ae_{k-1} + w_k)^T]$

$= E[Ae_{k-1}e_{k-1}^TA^T +$

$\cancel{Ae_{k-1}w_k^T} + \cancel{w_ke_{k-1}^TA^T}$

$+ w_kw_k^T]$

$= AE[e_{k-1}e_{k-1}^T]A^T + E[w_{k-1}w_{k-1}^T]$

$= AP_{k-1}A^T + Q$

• $P_k = P_k^- - K_kHP_k^- - P_k^-H^TK_k^T + K_kHP_k^-H^TK_k^T + K_kRK_k^T$

$= \cdots$ omitted $\cdots$

$= (I - K_kH)P_k^-$

In sum:

priori: $\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

cov.: $P_k^- = AP_{k-1}A^T + Q$ } PREDICT

posteriori: $\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k)$

cov.: $P_k = (I - K_kH)P_k^-$ $((I-KH)P^-(I-KH)^T + KRK^T)$

Kalman gain: $K_k = P_k^-H^T(HP_k^-H^- + R)^{-1}$