IEKF_for_ALAN_RPE_on_Manifold

given a dynamic & measurement model :

$$x_{k+1} = f(x_k, u_k, w_k)$$
$$y_k = h(x_k, v_k)$$

△ dynamic model

remark: Note that all states are in the non-inertial frame

△ $x \in R$

$$x = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} x_k + t u_k \\ y_k + t v_k \\ z_k + t w_k \\ \phi_k \\ \theta_k \\ \psi_k \\ u_k + tg(c\phi_k s\theta_k c\psi_k + s\phi_k s\psi_k) \\ v_k + tg(c\phi_k s\theta_k s\psi_k - s\phi_k c\psi_k) \\ w_k + tg(c\phi_k c\theta_k - 1) \end{bmatrix}, \quad F = \dots \text{ (omitted)}$$

$x \in \mathcal{M}$ (SE(3))

$$x = \begin{bmatrix} P \\ R \\ v \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} P_k + v_k \Delta t \\ R \\ v + R(g)t \end{bmatrix}, \quad F = \begin{bmatrix} \frac{\partial P_k}{\partial P} & \frac{\partial P_k}{\partial R} & \frac{\partial P_k}{\partial v} \\ \frac{\partial R_k}{\partial P} & \frac{\partial R_k}{\partial R} & \frac{\partial R_k}{\partial v} \\ \frac{\partial v_k}{\partial P} & \frac{\partial v_k}{\partial R} & \frac{\partial v_k}{\partial v} \end{bmatrix}$$

$\dot{x}\Delta t$

$\delta x = f'(x_k)$

$$= \begin{bmatrix} I_3 & 0 & I_3\Delta t \\ 0 & I_3 & 0 \\ 0 & -(Rg)_\times & I_3 \end{bmatrix}$$

$$J = \frac{\partial f(x)}{\partial x} \triangleq$$
$$\triangleq \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in R^{n \times m}$$
$$= \dots = \lim_{h \to 0} \frac{Jh}{h} = \frac{\partial Jh}{\partial h} = J$$

①
$$\therefore \frac{\partial v_k}{\partial R} = \frac{Rgt}{\partial R}$$
$$= \lim_{\theta \to 0} \frac{(R \oplus \theta) \cdot g - R \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(R \cdot Exp(\theta)) \cdot g - R \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(R \cdot (I + \theta_\times)) \cdot g - R \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{R\theta_\times \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{-R g_\times \theta}{\theta} \cdot t$$
$$= -R g_\times t \quad (J_R)$$

②
$$\therefore \frac{\partial v_k}{\partial R} = \frac{Rgt}{\partial R}$$
$$= \lim_{\theta \to 0} \frac{(\theta \oplus R) \cdot g - Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(Exp(\theta) \cdot R)g - Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(I + \theta_\times) Rg - Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{\theta_\times Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} -(Rg)_\times$$

△ measurement model

△ $y_{k+1} = h(x^-_{k+1})$, let $(P, R) = \xi$

$$\Rightarrow \begin{bmatrix} u_i \\ v_i \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} f_x \frac{X'_i}{Z'_i} + c_x \\ f_y \frac{Y'_i}{Z'_i} + c_y \\ \vdots \\ d_{k+1} - d_k \end{bmatrix} \Big\} \text{ 0 S ...}$$

(derived from $su = KTP$ $\quad d = T_{1:3,4}$
$\Rightarrow u = \frac{1}{s} KTP$)

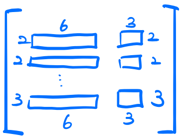
$$\therefore H = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial \xi} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial \xi} & \frac{\partial h_2}{\partial v} \\ \vdots & \vdots \\ \frac{\partial h_v}{\partial \xi} & \frac{\partial h_v}{\partial v} \end{bmatrix} = \begin{bmatrix} J_0 & 0 \\ J_1 & 0 \\ \vdots & \\ J_i & 0 \\ 0 & I_3 \end{bmatrix}$$

$[R^{2\times6}]$

$[R^{2\times9}]$

$$H = \begin{bmatrix} J_0 & 0' \\ J_1 & 0' \\ \vdots & \\ J_i & 0' \\ 0_{3\times6} & I_3 \end{bmatrix}, \quad J_i = -\begin{bmatrix} \frac{-f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X'Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & \frac{-f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & \frac{-f_y X'Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix} \in R^{2\times6}$$

$0' \in R^{2\times3}$

=


Iterated Extended Kalman filter

△ prediction same as EKF

△ correction different :

- after receive $\hat{x}_{k|k-1}$
- $\hat{x}^0_k = \hat{x}_{k|k-1}$ & start loop
- $H^i_k = \frac{\partial h(x)}{\partial x} \Big|_{x = \hat{x}^i_k}$
- $K^i_k = P_{k|k-1} H^{i\,T}_k (H^i_k P_{k|k-1} H^{i\,T}_k + R_k)^{-1}$
- $r^i_k = K^i_k (z_k - h(\hat{x}^i_k))$
- $\hat{x}^{i+1}_k = \hat{x}^i_k + r^i_k$
- break when $\|r^i_k\|_2 < \epsilon$

in our case, we have additional priori, thus residual slightly different

- embedd $\underbrace{Q, R}$ into one residual function
  ↳ determine K
  LHS

$$r = \begin{bmatrix} \alpha r_{mea} \\ \beta r_{odyn} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} [\begin{smallmatrix} u \\ v \end{smallmatrix}]_i \} - \frac{1}{s} K \hat{\chi}^i_k p \\ H' \hat{x}^0_k - H' \hat{x}^i_k \end{bmatrix}$$

- $\alpha = \frac{1}{tr(R)}$, $\beta = \frac{1}{tr(Q)}$
- $\frac{Dr}{D\chi} = J$, $\Delta x = -[JJ^T]^{-1} J f(x)$

Summary for ALAN-RPE

∴ for Prediction

$$\hat{\chi}^-_{k+1} = \chi_k \oplus \delta x$$
$$= \chi_k \circ Exp(\delta x) \quad \text{(euler's method)}$$
$$\hat{\chi}^-_{k+1} = RK_4(x_k, f'(x_k))$$
$$P^-_{k+1} = F_k P_k F_k + Q$$
$$F_k = \frac{Df}{D\chi}\Big|_{\chi = \chi_k}$$

∴ for Correction (IEKF)

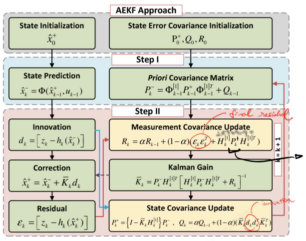
→ $\hat{\chi}^0_k := \hat{\chi}^-_k$

→ $r := \inf$

→ $i := 0$

→ while $r > \epsilon$

- $r = \begin{bmatrix} \alpha r_{mea} \\ \beta r_{odyn} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} [\begin{smallmatrix} u \\ v \end{smallmatrix}]_i \} - \frac{1}{s} K \hat{\chi}^i_k p \\ H' \hat{\chi}^0_k - H' \hat{x}^i_k \end{bmatrix}$
- $\alpha = \frac{1}{tr(R)}$, $\beta = \frac{1}{tr(Q)}$
- $\frac{Dr}{D\chi} = J$, $\Delta x = -[JJ^T]^{-1} J f(x)$
- $\hat{\chi}^{i+1}_k = \hat{\chi}^i_k \oplus \Delta x$



→ refer to stochastic modelling for GNSS paper

done in
↓
get Q first/

$$P_k = \begin{bmatrix} [SE(3)] & \\ & [v] \end{bmatrix}$$

$[R_{2\times2}]$ $[R_{3\times1}]$

$$\begin{bmatrix} [6\times6] & \\ & [3\times3] \end{bmatrix}$$

Summary : after optimization : use that for $H^{[1]}_k$.

△ Jacobian for a pinhole model during bundle adjustment

△ pinhole model :

$$s_i u_i = KTP_i$$

s_i — depth; K — Intrinsic; T — SE(3); P_i ~ 3D points

△ recall objective :

$$u_i - \frac{1}{s_i} KTP_i$$

intermediate variable

△ let $P' \triangleq$ PCD in camera frame

$$P' = (TP)_{1:3} = [X', Y', Z']^T$$

$$\therefore su = KTP$$
$$\Rightarrow sv = KP'$$
$$\Rightarrow \begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$
$$\therefore u = f_x \frac{X'}{Z'} + c_x$$
$$v = f_y \frac{Y'}{Z'} + c_y$$

△ $e = u - \frac{1}{s} KTP$

$$\frac{\partial e}{\partial \delta\xi} = \lim_{\delta\xi \to 0} \frac{e(\delta\xi \oplus \xi) - e(\xi)}{\delta\xi} = \underbrace{\frac{\partial e}{\partial P'}}_{1} \underbrace{\frac{\partial P'}{\partial \delta\xi}}_{2}$$

△
$$e = \begin{bmatrix} u - f_x \frac{X'}{Z'} + c_x \\ v - f_y \frac{Y'}{Z'} + c_y \end{bmatrix}$$

recall Jacobian:
$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

1
$$\therefore \frac{\partial e}{\partial P'} = \begin{bmatrix} \frac{\partial e_1}{\partial X'} & \frac{\partial e_1}{\partial Y'} & \frac{\partial e_1}{\partial Z'} \\ \frac{\partial e_2}{\partial X'} & \frac{\partial e_2}{\partial Y'} & \frac{\partial e_3}{\partial Z'} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \quad — ①$$

let $P' := [P'; 1]$

2
$$\frac{\partial P'}{\partial \delta\xi} = \frac{\partial Tp}{\partial \delta\xi}$$

Lie Groups
$$J = \frac{Df(\mathcal{X})}{D\mathcal{X}} = \lim_{\tau \to 0} \frac{f(\mathcal{X} \oplus \tau) \ominus f(\mathcal{X})}{\tau} \in R^{n\times m}$$
$$= \lim_{\tau \to 0} \frac{Log[f(\mathcal{X})^{-1} f(\mathcal{X} Exp(\tau))]}{\tau}$$

remark: cannot differentiate w.r.t. R, as R is a matrix. hence, use lie algebra on the local tangent plane

e.g.

$$= \lim_{\delta\xi \to 0} \frac{exp(\delta\xi^\wedge) exp(\xi^\wedge) p - exp(\xi^\wedge) p}{\delta\xi}$$
$$= \lim_{\delta\xi \to 0} \frac{(I + \delta\xi^\wedge) exp(\xi^\wedge) p - exp(\xi^\wedge) p}{\delta\xi}$$

$$\lim_{h \to 0} \frac{e^h e^x - e^x}{h} = \lim_{h \to 0} \frac{(1+h)e^x - e^x}{h} = \lim_{h \to 0} \frac{he^x}{h} = e^x$$

$$= \lim_{\delta\xi \to 0} \frac{\delta\xi^\wedge exp(\xi^\wedge) p}{\delta\xi}$$

$exp(\xi^\wedge) p = Tp = Rp + t$

$$= \lim_{\delta\xi \to 0} \frac{\begin{bmatrix} \delta\omega^\wedge & \delta\rho \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} Rp + t \\ 1 \end{bmatrix}}{\delta\xi}$$
$$= \lim_{\delta\xi \to 0} \frac{\begin{bmatrix} \delta\omega^\wedge (Rp + t) + \delta\rho \\ 0^T \end{bmatrix}}{\delta\xi}$$

$\delta\xi \rightsquigarrow \begin{bmatrix} \delta\rho \\ \delta\omega \end{bmatrix}$

recall Jacobian: $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

$$= \begin{bmatrix} \frac{\partial}{\partial \delta\rho}(\delta\omega^\wedge(Rp+t) + \delta\rho) & \frac{\partial}{\partial \delta\omega}(\delta\omega^\wedge(Rp+t + \delta\rho) \\ \frac{\partial}{\partial \delta\rho} \; 0 & \frac{\partial}{\partial \delta\omega} \; 0 \end{bmatrix}$$
$$= \begin{bmatrix} I & -(Rp+ \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I & -P'^\wedge \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

- Here $P' = [x', y', z']$

$$\therefore \frac{\partial P'}{\partial \delta\xi} = [I \;\; -P'^\wedge]$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$$

- as $\frac{\partial e}{\partial P'} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix}$

& $\frac{\partial P'}{\partial \delta\xi} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$

$$\therefore \frac{\partial e}{\partial \delta\xi} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X'Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & \frac{-f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & \frac{-f_y X'Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix}$$