- optimization problem

$$\underset{x}{\text{minimize}}\ f_0(x)$$
$$\text{s.t.}\quad f_i(x) \le 0,\quad h_i(x) = 0$$

- feasible
  $x \in \text{dom } f_0$
- infeasible
  $p^* = \infty / -\infty$
- optimal value
  $p^* = \min\{f_0(x) \mid f_i(x) \le 0 .\ h_i(x) = 0\}$
  x^* is optimal
- locally optima
  $f_0(x) = \inf\{f_0(z) \mid z \in D.\ \|z-x\|_2 \le R\}$
  $D = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=0}^{p} \text{dom} h_i$
- feasibility problem

$$\text{find } x \quad \text{s.t. } f_i(x) \le 0,\ h_i(x) = 0 \iff \underset{x}{\text{minimize}}\ 0 \quad \text{s.t. } f_i(x) \le 0,\ h_i(x) = 0$$

- convex optimization

$$\underset{x}{\text{minimize}}\ f_0(x) \rightsquigarrow \text{convex}$$
$$\text{s.t.}\quad f_i(x) \le 0 \rightarrow \text{convex},\quad a_i^T x = b_i \rightarrow \text{affine}$$

- quasiconvex optimization

$$\underset{x}{\text{minimize}}\ f_0(x) \rightsquigarrow \text{quasiconvex}$$
$$\text{s.t.}\quad f_i(x) \le 0,\quad a_i^T x = b_i$$

$S_\alpha = \{x \in D \mid f(x) \le \alpha\}$
$\alpha = R$
$S_\alpha = \text{convex}$
$f(x) \in \text{quasiconvex}$

- local & global optima

$$\underset{z}{\text{minimize}}\ f_0(z)$$
$$\text{s.t.}\quad f_i(z) \le 0,\quad h_i(z) = 0,\quad \|z-x\|_2 \le R$$

x is locally optimal when
$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z-x\|_2 \le R\}$

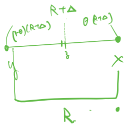
fall back here

in convex
if x is local optimal
then x is global optimal
proof:

△ if $\exists y$ $\left.\begin{array}{l} f_0(y) < f_0(x) \\ \|y-x\|_2 > R \end{array}\right\}$ as if $\|y-x\|_2 < R$

△ we let
$$\theta = \frac{R}{2\|y-x\|_2} < \frac{1}{2}$$

△ we let
$$z = (1-\theta)x + \theta y$$
convex combination, in convex function
↳ ∴
$$f_0(z) \le (1-\theta) f_0(x) + \theta f_0(y)$$
$$0 < \theta < \frac{1}{2}$$



⇒ from here
$\|x-z\|_2 = \theta(R+\Delta)$
$\le \frac{1}{2}(R+\Delta)$

- we let $\|x-z\|_2 = \frac{1}{2}R$ to fullfil all the condition (we want z to be within $\|x-z\| < R$, s.t. x is local optimal)

△ w/ assumption $f_0(y) < f_0(x)$
we also hv
$$\theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$$
$$\Rightarrow f_0(z) = \theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$$
↳ contradicts w/
$$f_0(x) = \inf\{f_0(z) \mid z \in D.\ \|z-x\|_2 \le R\}$$

- optimality criterion
$$\nabla f_0(x)^T (y-x) \ge 0 \quad \forall y \in D$$

- Convex problems
  - Linear Programming

$$\underset{x}{\text{minimize}}\ c^T x + d$$
$$\text{s.t.}\quad Gx \preceq h,\quad Ax = b$$

  - piecewise-linear minimization

$$f_0(x) = \max_{i=1,\dots,m} (a_i^T x + b_i)$$
$$\underset{x}{\text{minimize}}\ f_0(x)$$
$$\Updownarrow$$
$$\text{minimize}\ t \quad \text{s.t.}\ a_i^T x + b_i \le t \quad i = 1, \dots, m$$

  - Quadratic Program

$$\underset{x}{\text{minimize}}\ \frac{1}{2} x^T P x + q^T x + r$$
$$\text{s.t.}\quad Gx \preceq h,\quad Ax = b$$
$$P \in S_+^n$$

  - Quadratically Constrained Quadratic Program

$$\underset{x}{\text{minimize}}\ \frac{1}{2} x^T P_0 x + q_0^T x + r_0$$
$$\text{s.t.}\quad \frac{1}{2} x^T P_i x + q_i^T x + r_i \le 0,\quad Ax = b$$
$$P_i \in S_+^n$$
(cover a feasible region that's intersection of ellipsoids + affine set)

- Geometric Programming
  - monomial function
    $f: R^n \to R$ $\text{dom} f = R_{++}^n$
    $f(x) = c x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$
    $c > 0.\ a_i \in R$
  - posynomial function
    $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$
    $c_k > 0,\ a_{ik} \in R$
    sum of monomial function
  - transfer to convex problem
    $y_i = \log x_i \quad x_i = e^{y_i}$

$$f(x) = f(e^{y_1}, \dots, e^{y_n}) = c(e^{y_1})^{a_1} \dots (e^{y_n})^{a_n} = \underbrace{e^{a^T y + b}}_{\text{monomial}} \quad (b = \log c)$$

original

$$\text{minimize}\ f_0(x) \quad \text{s.t.}\ f_i(x) \le 1,\ h_i(x) = 1$$
mono, posy

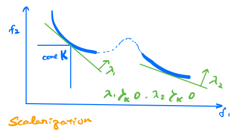
convex equivalence

$$\text{minimize}\ \log\left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}}\right)$$
$$\text{s.t.}\ \log\left(\sum_{k=1}^{K} e^{a_{ik}^T y + b_{ik}}\right) \le 0,\quad Gy + d = 0$$

- multicriterion optimization

$$\text{minimize}\ f_0(x) = [F_1(x), \dots F_q(x)]$$
$$\text{s.t.}\quad f_i(x) \le 0,\quad Ax = b$$

  - Pareto optimal
    $(f_0(x) - K) \cap = f_0(x)$



  - Scalarization

$$\text{minimize}\ \lambda^T f_0(x) \quad \text{s.t.}\ f_i(x) \le 0,\ h_i(x) = 0$$

$\Rightarrow \forall x^{PO},$
$\exists \lambda \succ_{K^*} 0$
s.t. $\tilde{x}$ is an optimal point of the scalarized problem

- Second-order Cone Programming

$$\underset{x}{\text{minimize}}\ f^T x$$
$$\|A_i x + b_i\|_2 \le c_i^T x + d_i$$
$$Fx = g$$

SOCP could be understood:
$\begin{bmatrix} Ax+b \\ c^T x + d \end{bmatrix} \in R^{k+1}$ lies in SOC
$(Ax+b, c^T x + d) \succeq_Q 0$
G.I. for cone Q

- e.g. Robust Linear Programming

- Duality
  - The Lagrangian

$$\text{minimize}\ f_0(x) \rightarrow \lambda_i$$
$$\text{s.t.}\quad f_i(x) \le 0 \quad i = 1, \dots, m$$
$$h_i(x) = 0 \quad i = 1, \dots, p \rightarrow \nu_i$$
$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

  - The Lagrange Dual Function
    $g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$
    ∴ $\underbrace{g(\overset{\succeq 0}{\lambda}, \nu)}_{\text{lower bound}} \le p^* = f_0(\tilde{x}) \rightarrow$ all feasible x
    ∴ $\underbrace{\lambda \succeq 0, (\lambda, \nu) \in \text{dom } g}_{\text{dual feasible}}$
  - The Lagrange Dual Problem

$$\text{maximize}\ g(\lambda, \nu) \quad \text{s.t.}\ \lambda \succeq 0$$

- Weak Duality
  $d^* \le p^*$
  best bound, optimal value
  $\Delta =$ duality gap
- Strong Duality
- $d^* = p^*$
- slater's condition
  $\exists x \in \text{relint } D$
  s.t.
  $f_i(x) < 0$, $Ax = b$

- optimality conditions — complementary slackness
  assume strong duality holds:
  $d^* = p^*$
  $g(\lambda^*, \nu^*) = f_0(x^*)$

⇒

$$f_0(x^*) = g(\lambda^*, \nu^*) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)\right)$$
$$\le f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^*}_{\ge 0} \underbrace{f_i(x^*)}_{\le 0} + \sum_{i=1}^{p} \nu_i^* \underbrace{h_i(x^*)}_{=0}$$
$$\le f_0(x^*)$$

$\begin{cases} \text{if } \lambda_i^* > 0 \Rightarrow f_i(x^*) = 0 \\ \text{if } f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0 \end{cases}$
complementary slackness!

∴ $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$

- KKT optimality conditions
  - let x^* (primal), (λ^*, ν^*) (dual) be optimal points
  - 0-duality gap

∴ $x^* = \underset{x}{\text{argmin}}\ L(x, \lambda^*, \nu^*)$

- $\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$
  ∴
- $f_i(x^*) \le 0$
  $h_i(x^*) = 0$
  $\lambda_i^* \ge 0$
  $\lambda_i^* f_i(x^*) = 0$

→ KKT optimality conditions!

- Perturbation & Sensitivity Analysis
- recall

$$\underset{x}{\text{minimize}}\ f_0(x) \quad \text{s.t.}\ f_i(x) \le 0,\ h_i(x) = 0$$

perturbed by $u_i . v_i$

$$\underset{x}{\text{minimize}}\ f_0(x) \quad \text{s.t.}\ f_i(x) \le u_i,\ h_i(x) = v_i$$

- optimal value (becomes a function) w.r.t. (u, v)
  $p^*(u, v) = \inf\left\{f_0(x) \,\middle|\, \begin{array}{l} \exists x \in D \\ f_i(x) \le u_i \\ h_i(x) = v_i \end{array}\right\}$
- let $(\lambda^*, \nu^*) \rightarrow$ dual optimal pt. of unperturbed problem

$$\Rightarrow p^*(0,0) = g(\lambda^*, \nu^*) \le f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \le f_0(x) + \lambda^{*T} u + \nu^{*T} v$$
$$\Rightarrow p^*(0,0) \le f_0(x) + \lambda^{*T} u + \nu^{*T} v$$
$$\Rightarrow f_0(x) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$$
$$\Rightarrow p^*(u,v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$$
$(p^*(u,v) \in \{f_0(x) \mid x \in \text{dom} f\})$

$$p^*(u,v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$$
perturbation analysis

- Sufficiency & Necessity

if P then Q

△ Q is necessary for P
① truth of Q is guaranteed by truth of P
② (falsity of Q guarantees falsity of P)

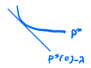
▲ P is sufficient for P
③ truth of P guarantees truth of Q
④ but falsity of P might still hv truth of Q

如果"要选总统"$_P$ 那"需40yo"$_Q$
e.g.
① "人一定是40岁"$_Q$ 如果"这人选总统"$_P$
② 如果"未40yo"$_Q$ 那"一定非总统"$_P$
③ 如果"要选总统"$_P$ 那"一定40yo"$_Q$
④ 如果"不选总统"$_P$ 那"可能40yo"$_Q$

sum:
if P then Q
△ Q is P's necessary condition
▲ P is Q's sufficient condition

- local sensitivity
  assume
  - $p^*(u,v)$ differentiable @ $u=0$, $v=0$
  - strong duality holds

⇒ $p^*(u,v)$ @ $u=0, v=0$
$$\frac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$$



$$\frac{\partial p^*(0,0)}{\partial v_i} = -\nu_i^*$$

- w/ complementary slackness:
  $\lambda_i f_i(x) = 0$
  - $f_i(x) < 0$, $\lambda_i = 0$
    then: value @ $p^*(0,0)$ doesn't change w.r.t. u
  - $f_i(x) = 0$, $\lambda_i^* > 0$
    then: value @ $p^*(0,0)$ changes w.r.t. u
    $\lambda_i^* \uparrow \rightarrow$ perturb ↑
    $\lambda_i^* \downarrow \rightarrow$ perturb ↓

→ tells us HOW ACTIVE IT IS

- **Numerical Method of Differential Equations**

- **ODE 4 types**

1 seperable equations

$$\frac{dy}{dx}=P(x)Q(x)$$

e.g. $y'+2xy=x$

$\Rightarrow \frac{dy}{dx}+2xy=x$

$\Rightarrow \frac{dy}{dx}=x-2xy$

$\Rightarrow \frac{dy}{dx}=x(1-2y)$

$\Rightarrow dy=x(1-2y)dx$

$\Rightarrow \frac{1}{1-2y}dy=x\,dx$

$\Rightarrow \int\frac{1}{1-2y}dy=\int x\,dx$

$\Rightarrow -\frac{1}{2}\ln(1-2y)=\frac{1}{2}x^2+C$

$\Rightarrow e^{-\frac{1}{2}\ln(1-2y)}=e^{-\frac{1}{2}x^2+C}$

$\Rightarrow 1-2y=ce^{-x^2}$

$\Rightarrow y=-\frac{ce^{-x^2}-1}{2}$

2. homogenous method

$$f(kx,ky)=f(x,y)$$

e.g.

$\frac{dy}{dx}=\frac{x^2+y^2}{xy}$

check $\frac{k^2x^2+k^2y^2}{kxky}=\frac{x^2+y^2}{xy}$

let $v=\frac{y}{x}$ $\frac{1}{v}=\frac{x}{y}$

$y=vx$

$\frac{dy}{dx}=x\frac{dv}{dV}+v$

- $\frac{dy}{dx}=\frac{\frac{x^2}{x^2}+\frac{y^2}{x^2}}{\frac{xy}{x^2}}=\frac{1+(\frac{y}{x})^2}{\frac{y}{x}}$

$=\frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{1}{v}+v$

$\Rightarrow x\frac{dv}{dx}=\frac{1}{v}$

$\Rightarrow v\,dv=\frac{1}{x}dx$

$\Rightarrow \int v\,dv=\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}v^2=\ln(x)+C$

$\Rightarrow v^2=2\ln|x|+C$

$v=\pm\sqrt{2\ln|x|+C}$

$\frac{y}{x}=\pm\sqrt{2\ln|x)+C}$

$y=\pm x\sqrt{2\ln|x|+C}$

3. Integrating factor

$$\frac{dy}{dx}+P(x)y=Q(x)$$

e.g. $\frac{dy}{dx}+1y=x$
$\quad P(x)=1$

$\Leftrightarrow \frac{dy}{dx}+P(x)y=1$ ①

$\mu(x)=e^{\int P(x)dx}$

$\mu(x)=e^{\int 1dx}$

$=e^x$ ②

$\Rightarrow$ ① & ②

$$\mu(x)\left[\frac{dy}{dx}+P(x)y=Q(x)\right]$$

$\Rightarrow e^x\left[\frac{dy}{dx}+y=x\right]$

$\Rightarrow e^x\frac{dy}{dx}+ye^x=xe^x$

$\Rightarrow \frac{d}{dx}(e^xy)=xe^x$

$\Rightarrow \int\frac{d}{dx}e^xy=\underbrace{\int xe^xdx}_{IBP}$

$\Rightarrow e^xy=xe^x-e^x+C$

$\Rightarrow y=x-1+\frac{C}{e^x}$

$=x-1+Ce^{-x}$

- $\frac{dv}{dt}=g-\frac{c}{m}v$

$$\frac{dv}{dt}+P(t)v=Q(t)$$

$\Rightarrow \frac{dv}{dt}+(\frac{c}{m})v=g$

$\Rightarrow \mu(t)=e^{\int\frac{c}{m}dt}$

$=e^{\frac{c}{m}t}$

$\Rightarrow e^{\frac{c}{m}t}\left(\frac{dv}{dt}+(\frac{c}{m})v=g\right)$

$\Rightarrow e^{\frac{c}{m}t}\frac{dv}{dt}+\frac{c}{m}ve^{\frac{c}{m}t}=ge^{c/mt}$

$\Rightarrow \frac{d}{dt}e^{\frac{c}{m}t}v=ge^{c/mt}$

$\Rightarrow \int\frac{d}{dt}e^{c/mt}v=\int ge^{c/mt}dt$

$\Rightarrow e^{c/mt}v=\frac{m}{c}ge^{\frac{c}{m}t}+c'$

$\Rightarrow v=e^{-c/mt}\left[\frac{m}{c}ge^{\frac{c}{m}t}+c'\right]$

assume $v(0)=0$

$v(0)=+\frac{m}{c}g+c'=0$

$c'=\frac{-m}{c}g$

$\therefore v(t)=e^{-c/mt}\left[\frac{m}{c}ge^{-\frac{c}{m}t}-\frac{m}{c}g\right]$

$=e^{-c/mt}\frac{m}{c}g\left[e^{-\frac{c}{m}t}-1\right]$

$=\frac{mg}{c}\left[1-e^{-c/mt}\right]$

- **linear ODE**
  - Laplace
  - analytically solved
  - $a_0(x)y+a_1(x)y'+\dots+a_n(x)y^{(n)}=b(x)$
  - $Ly=f$

- **Runge - Kutta Methods**

△ Taylor series

$\frac{dy}{dx}=f(x,y)$

$\frac{d^{(n)}y}{dx^{(n)}}=f^{(n-1)}(x,y)$

$y_{i+1}=\underline{y_i+f(x_i,y_i)h}$

Euler's Method

△ $y_{i+1}=y_i+f(x_i,y_i)h+\frac{f'(x_i,y_i)}{2!}h^2+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^n+O(h^{n+1})$

$\simeq y_i+\left[f(x_i,y_i)+\frac{f'(x_i,y_i)}{2!}h+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}\right]h$

$=y_i+\phi(x_i,y_i,h)h$

△ $\phi(x_i,y_i,h)$

$=f(x_i,y_i)+\frac{f'(x_i,y_i)}{2!}h+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}$

△ Runge-Kutta :

acquire $\phi(x_i,y_i,h)$

w.o./ calculating higher-order term

$y_{i+1}=y_i+\phi(x_i,y_i,h)h$

$\Rightarrow \phi(x_i,y_i,h)$

$=a_1k_1+a_2k_2+\dots+a_{n-1}k_{n-1}+a_nk_n$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+p_1h,\ y_i+q_{11}k_1h)$

$k_3=f(x_i+p_2h,\ y_i+q_{21}k_1h+q_{22}k_2h)$

$\vdots$

$k_n=f(x_i+p_{n-1}h,\ y_i+q_{n-1,1}k_1h+q_{n-1,2}k_2h+q_{n-1,3}k_3h+\dots+q_{n-1,n-1}k_{n-1}h)$

$a,p,q\Rightarrow$ constants

$n=$ order $\Rightarrow$ RK n method !

e.g.

△ $y=-0.5x^4+4x^3-10x^2+8.5x+1$

$\frac{dy}{dx}=-2x^3+12x^2-20x+8.5$

$f(0)=1$

$f(0.5)=?$

△ RK4 :

$y_{i+1}=y_i+\phi(x_i,y_i,h)h$

$\phi(x_i,y_i,h)$

$=\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_1h)$

$k_3=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_2h)$

$k_4=f(x_i+h,\ y_i+k_3h)$

△ $\therefore$

$k_1=f(0,1)=8.5$

$k_2=f(0+0.25,1+0.5\cdot8.5\cdot0.5)=4.21875$

$k_3=f(0+0.25,1+0.5\cdot4.21875\cdot0.5)=4.21875$

$k_4=f(0.5,1+4.21875\cdot0.5)=1.25$

$y(0.5)=1+\frac{0.5}{6}(8.5+2\cdot4.21875+2\cdot4.21875+1.25)$

$=3.21875+O(h^5)$

- **system of equations**

$$\begin{cases}\frac{dy_1}{dx}=f_1(x,y_1,y_2,\dots,y_n)\\ \frac{dy_2}{dx}=f_2(x,y_1,y_2,\dots,y_n)\\ \vdots\\ \frac{dy_n}{dx}=f_n(x,y_1,y_2,\dots,y_n)\end{cases}$$

e.g.

△ $\begin{cases}\frac{dy_1}{dx}=-0.5y_1\\ \frac{dy_2}{dx}=4-0.3y_2-0.1y_1\end{cases}$

$y_1=4$, $y_2=6$ @ $x=0$

△ $\dot{y}=f(x)$

$$=\begin{bmatrix}\frac{dy_1}{dx}\\ \frac{dy_2}{dx}\end{bmatrix}=\begin{bmatrix}-0.5&0\\-0.1&-0.3\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}+\begin{bmatrix}0\\4\end{bmatrix}$$

△ RK4

$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_1h)$

$k_3=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_2h)$

$k_4=f(x_i+h,\ y_i+k_3h)$

△ $k_{1,1}=f_1(0,4)=-0.5\cdot4=-2$

$k_{1,2}=f_2(0,6)=4-0.3\cdot6-0.1\cdot4=1.8$

△ $k_{2,1}=f_1(0+\frac{1}{2}\cdot0.5,\ 4+\frac{1}{2}\cdot(-2)\cdot0.5)$

$=-0.5\cdot3.5=-1.75$

$k_{2,2}=f_2(0+\frac{1}{2}0.5,\ 6+\frac{1}{2}\cdot(1.8)\cdot0.5)$

$=4-0.3\cdot6.45-0.1\cdot3.5$

$=1.715$

△ $k_{3,1}=f_1(0+\frac{1}{2}\cdot0.5,\ 4+\frac{1}{2}(-1.75)0.5)$

$=-0.5\cdot3.5625$

$=-1.78125$

$k_{3,2}=f_1(0+\frac{1}{2}\cdot0.25,\ 6+\frac{1}{2}(1.715)\cdot0.5)$

$=4-0.3\cdot6.42875-0.1\cdot3.5625$

$=1.715125$

△ $k_{4,1}=f_1(0+0.5,\ 4+(-1.78125)\cdot0.5)$

$=-0.5\cdot3.109375$

$=-1.5546875$

$k_{4,2}=f_2(0+0.5,\ 6+(1.715125\cdot0.5))$

$=4-0.3\cdot6.8575625-0.1\cdot3.109375$

$=1.63179375$

△ $y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$y=\begin{bmatrix}4\\6\end{bmatrix}+\frac{0.5}{6}\left(\begin{bmatrix}-2\\1.8\end{bmatrix}+2\begin{bmatrix}-1.75\\1.715\end{bmatrix}+2\begin{bmatrix}-1.78125\\1.715125\end{bmatrix}+\begin{bmatrix}-1.5546875\\1.63179375\end{bmatrix}\right)$

$=\begin{bmatrix}3.115234\\6.875670\end{bmatrix}$ #

- **solving high-order ODE**

$$\begin{cases}\frac{d^2y}{dx^2}+2\frac{dy}{dx}=x+y\\ y=1\quad\frac{dy}{dx}=2\quad\text{when }x=0\end{cases}$$

$\begin{cases}\text{let}\quad u=\frac{dy}{dx}\\ \frac{du}{dx}=\frac{d^2y}{dx^2}\end{cases}$

$\begin{cases}\Rightarrow\frac{du}{dx}+2u=x+y\\ \Rightarrow\frac{du}{dx}=x+y-2u\end{cases}$

$\begin{cases}\frac{dy}{dx}=u\\ \frac{du}{dx}=x+y-2u\end{cases}$ $\quad y(0)=1$, $u(0)=2$

use this to propagate

$$\begin{cases}\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+3\frac{dy}{dx}=x+y\\ y=1\quad\frac{dy}{dx}=2\quad\frac{d^2y}{dx^2}=3\quad x=0\end{cases}$$

$\begin{cases}\frac{dy}{dx}=u\\ \frac{d^2y}{dx^2}=v\end{cases}$ $\quad\frac{d^2y}{dx^2}=\frac{du}{dx}=v$, $\quad\frac{d^3y}{dx^3}=\frac{dv}{dx}$

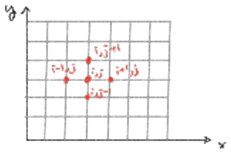
$\begin{cases}\frac{dv}{dx}+2v+3u=x+y\\ \frac{dy}{dx}=u\\ \frac{du}{dx}=v\end{cases}$ $\quad y(0)=1$, $u(0)=2$, $v(0)=3$, $x=0$

use this to propagate

- **PDE**

- Laplace equation

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$$



$\frac{\partial^2T}{\partial x^2}=\frac{\left(\frac{T_{i+1,j}-T_{i,j}}{\Delta x}\right)-\left(\frac{T_{i,j}-T_{i-1,j}}{\Delta x}\right)}{\Delta x}$

$=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}$

$\frac{\partial^2T}{\partial y^2}=\frac{\left(\frac{T_{i,j+1}-T_{i,j}}{\Delta y}\right)-\left(\frac{T_{i,j}-T_{i,j-1}}{\Delta y}\right)}{\Delta y}$

$=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}$

$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$

$\Rightarrow \frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=0$

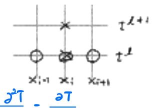
as $\Delta x=\Delta y$

$\Rightarrow T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=0$

$\Rightarrow T_{i,j}=\frac{T_{i+1,j}+T_{i-1,j}+T_{i,j+1},T_{i,j-1}}{4}$

- Parabolic Equation

$k\frac{\partial^2T}{\partial x^2}=\frac{\partial T}{\partial t}$



$k\frac{\partial^2T}{\partial x^2}=\frac{\partial T}{\partial t}$

$\frac{\partial^2T}{\partial x^2}=\frac{\frac{T^l_{i+1}-T^l_i}{\Delta x}-\frac{T^l_i-T^l_{i-1}}{\Delta x}}{\Delta x}$

$\frac{\partial T}{\partial t}=\frac{T^{l+1}_i-T^l_i}{\Delta t}$

$\Rightarrow k\frac{T^l_{i+1}-2T^l_i+T^l_{i-1}}{\Delta x^2}=\frac{T^{l+1}_i-T^l_i}{\Delta t}$

$\Rightarrow T^{l+1}_i-T^l_i=\frac{k\Delta t}{\Delta x^2}(T^l_{i+1}-2T^l_i+T^l_{i-1})$

$\Rightarrow T^{l+1}_i=T^l_i+\lambda(T^l_{i+1}-2T^l_i+T^l_{i-1})$

$\lambda=\frac{k\Delta t}{\Delta x^2}$

e.g.

$\Delta x=2$ cm

$\Delta t=0.1$ s

$k=0.835$ cm^2/s

$\lambda=0.020875$

$$\begin{cases}T(\overset{\text{position}}{0})=100 & @\ \forall t\\ T(10)=50\\ T(x)=0 & @\ t=0,\ 0<x<10\end{cases}$$

$T^{l+1}_i=T^l_i+\lambda(T^l_{i+1}-2T^l_i+T^l_{i-1})$

△ compute T @ $t=0.1$ s, $x=2$ cm

$T^1_1=T^{\overset{\text{time}}{0}}_{\underset{\text{position}}{1}}+0.020875(T^0_2-2T^0_1+T^0_0)$

$=0+0.020875(0-2\cdot0+100)$

$=2.0875$

△ compute T @ $t=0.1$s, $x=4$ cm

$T^1_2=T^0_2+0.020875(T^0_3-2T^0_2+T^0_1)$

$=0+0.020875(0-2\cdot0+0)$

$=0$

△ compute T @ $t=0.1$s, $x=6$cm

$T^1_3=T^0_3+0.020875(T^0_4-2T^0_3+T^0_2)$

$=0+0.020875(0-2\cdot0+0)$

$=0$

△ compute T @ $t=0.1$s, $x=8$ cm

$T^1_4=T^0_4+0.020875(T^0_5-2T^0_4+T^0_3)$

$=0+0.020875(50-0+0)$

$=1.0438$

△ Compute T @ $t=0.2$s, $x=2$cm

$T^2_1=T^1_1+\lambda(T^1_2-2T^1_1+T^1_0)$

$=2.0875+0.020875(0-2\cdot2.0875+100)$

$=4.0878$

- **convergence**

$\lambda=\frac{k\Delta t}{\Delta x^2}\leqslant\frac{1}{2}\Rightarrow\Delta t\leq\frac{1}{2}\frac{\Delta x^2}{k}$

- $\frac{\partial T}{\partial t}=k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)$

$$\begin{cases}\frac{\partial^2T}{\partial x^2}=\frac{T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j}}{\Delta x^2}\\ \frac{\partial^2T}{\partial y^2}=\frac{T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1}}{\Delta y^2}\\ \frac{\partial T}{\partial t}=\frac{T^{l+1}_{i,j}-T^l_{i,j}}{\Delta t}\end{cases}$$

$k\left(\frac{T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j}}{\Delta x^2}+\frac{T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1}}{\Delta y^2}\right)$

$=\frac{T^{l+1}_{i,j}-T^l_{i,j}}{\Delta t}$

$\Rightarrow T^{l+1}_{i,j}=T^l_{i,j}+\frac{k\Delta t}{\Delta x^2}(T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j})+\frac{k\Delta t}{\Delta y^2}(T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1})$

let $\Delta x=\Delta y$ $\quad\lambda=\frac{k\Delta t}{\Delta x^2}=\frac{k\Delta t}{\Delta y^2}$

$\Rightarrow T^{l+1}_{i,j}=T^l_{i,j}+\lambda(T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j}+T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1})$

- **convergence**
- 1- dimension :

$\lambda=\frac{k\Delta t}{\Delta x^2}\leq\frac{1}{2}$

- 2-dimension :

$\Delta t\leq\frac{1}{8}\frac{\Delta x^2+\Delta y^2}{k}$

$\Downarrow$

if $\Delta x^2=\Delta y^2$

$\Delta t\leq\frac{1}{4}\frac{\Delta x^2}{k}$

- **Linear Programming**
  - Example problem

- a_1 origin 1   costs c_{ij} terminal 1 b_1

  a_2 origin 2   terminal 2 b_2 terminal 3 b_3

- $$\underset{x}{\text{minimize}}\ [c_{11}\ c_{12}\ c_{13}]\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\end{bmatrix}+[c_{21}\ c_{22}\ c_{23}]\begin{bmatrix}x_{21}\\x_{22}\\x_{23}\end{bmatrix}$$

$$\text{s.t.}\ \begin{cases}x_{11}+x_{12}+x_{13}=a_1\\x_{21}+x_{22}+x_{23}=a_2\end{cases}\quad\begin{cases}x_{11}+x_{21}=b_1\\x_{12}+x_{22}=b_2\\x_{13}+x_{23}=b_3\end{cases}$$

$$x_{ij}\geq 0$$

$$\Rightarrow\ \underset{x}{\text{minimize}}\ [c_{11}\ c_{12}\ c_{13}\ c_{21}\ c_{22}\ c_{23}]\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{21}\\x_{22}\\x_{23}\end{bmatrix}$$

s.t.

$$\begin{bmatrix}1&1&1&0&0&0\\0&0&0&1&1&1\\1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix}\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{21}\\x_{22}\\x_{23}\end{bmatrix}=\begin{bmatrix}a_1\\a_2\\b_1\\b_2\\b_3\end{bmatrix}$$

$$x\succeq 0$$

$$\Rightarrow\ \underset{x}{\text{minimize}}\ c^Tx \qquad \text{s.t.}\ Ax=b,\ x\succeq 0$$

- transform to standard form

$$\underset{x}{\text{minimize}}\ c^Tx\quad \text{s.t.}\ ax\leq b,\ x\succeq 0 \quad\Updownarrow\quad \underset{x.y}{\text{minimize}}\ c^Tx\quad \text{s.t.}\ ax+y=b,\ x\succeq 0,\ y\succeq 0$$

$$\underset{x}{\text{minimize}}\ c^Tx\quad \text{s.t.}\ ax\geq b,\ x\succeq 0 \quad\Updownarrow\quad \underset{x.y}{\text{minimize}}\ c^Tx\quad \text{s.t.}\ ax-y=b,\ x\succeq 0,\ y\succeq 0$$

$$\underset{x.y}{\text{minimize}}\ c^Tx+y \quad \text{s.t.}\ Ax+y=b,\ x\succeq 0\ (\text{no } y\succeq 0)$$

$\Updownarrow$ let $y=u-v$

$$\underset{x.y.u.v}{\text{minimize}}\ c^Tx+u-v \quad Ax+u-v=b \quad x\succeq 0\ u\succeq 0\ v\succeq 0$$

$$\underset{x.y.z}{\text{minimize}}\ c^Tx+y+z \quad \text{s.t.}\ Ax+y+z=b,\ y+z=d,\ x\succeq 0\ y\succeq 0\ (\text{no } z\succeq 0)$$

$\Updownarrow$ let $z=b-Ax-y$

$$\underset{x\,y}{\text{minimize}}\ c^Tx+y+b-Ax-y \quad \text{S.T.}\ y+b-Ax-y=d,\ x\succeq 0\ y\succeq 0$$

$\Updownarrow$

$$\underset{x\,y}{\text{minimize}}\ c^Tx+b-Ax \quad \text{S.T.}\ b-Ax=d,\ x\succeq 0\ y\succeq 0$$

- Solution –
  Simplex method
  高中線性歸化
- Solution –
  Basic Feasible solution (BFS)

$$\underset{x}{\text{minimize}}\ c^Tx \quad \text{s.t.}\ Ax=b,\ x\succeq 0$$

let $\tilde{x}$ be
1. $\tilde{x}\in D$, i.e., $A\tilde{x}=b$, $\tilde{x}\succeq 0$
2. cols of A corresponding to non-zero of $\tilde{x}$ are linearly independent

Theorem: $\tilde{x}$ is an extreme (rank(A)=m) i.f.f it is BFS

1. Basic/non-Basic variables
   - for any BFS $\tilde{x}$, we rewrite:

$$\tilde{x}'=\begin{pmatrix}x_B\\x_N\end{pmatrix}\quad B:\text{non-zero}\qquad A=(B,N)\quad N:\text{zero}\qquad c=\begin{pmatrix}c_B\\c_N\end{pmatrix}$$

   - $$\left.\begin{aligned}&Ax=b\\&\Updownarrow\\&Bx_B+Nx_N=b\\&\Updownarrow\\&Bx_B=b\end{aligned}\right\}\ \begin{aligned}&\text{rank}(B)=m_b\\&x_B=B^{-1}b\\&x_N=0^{n-m_b}\end{aligned}$$
   - $Bx_B+Nx_N=b$
     $$\begin{cases}x_B=B^{-1}(b-Nx_N)\geq 0^{m_b}\\ x_N\geq 0^{n-m_b}\end{cases}$$

2. $$\underset{x}{\text{minimize}}\ c^Tx\ \ \text{s.t.}\ Ax=b,\ x\succeq 0^n \iff \underset{x}{\text{minimize}}\ c_B^Tx_B+c_N^Tx_N\ \ \text{s.t.}\ Bx_B+Nx_N=b,\ x_B\succeq 0^{m_b},\ x_N\succeq 0^{n-m_b}$$

$$\iff \underset{x}{\text{minimize}}\ c_B^T(B^{-1}(b-Nx_N))+c_N^Tx_N \quad \text{s.t.}\ B^{-1}(b-Nx_N)\succeq 0^{m_b},\ x_N\succeq 0^{n-m_b}$$

- **Simplex method: the Tableau**

$$\text{minimize}\ x_1-x_2 \quad \text{s.t.}\ x_1-x_2\leq 2,\ x_1+x_2\leq 6,\ x_1\geq 0,\ x_2\geq 0$$

$x_1=0$
$x_2=6$
$x_3=8$
$x_4=0$

$\Updownarrow$

$$\text{minimize}\ p=x_1-x_2 \quad \text{S.T.}\ x_1-x_2+s_1=2,\ x_1+x_2+s_2=6,\ x_1\geq 0\ x_2\geq 0,\ s_1\geq 0\ s_2\geq 0$$

$$-x_1+x_2=p$$
$$-p-x_1+x_2=0$$
$$\Rightarrow p+x_1-x_2=0$$

$$\begin{array}{ccccc|c|c} x_1&x_2&s_1&s_2&p&&\\ 1&-1&1&0&0&2&-2\\ 1&1&0&1&0&6&6\\ \hline 1&-1&0&0&1&0& \end{array}$$

$\Updownarrow$

$$\begin{bmatrix}2&0&1&1&0&8\\1&1&0&1&0&6\\ \hline 1&-1&0&0&1&0\end{bmatrix}$$

$\Updownarrow$

$$\begin{bmatrix}2&\boxed{0}&1&1&0&\boxed{8}\\1&1&0&1&0&6\\2&0&0&1&1&6\end{bmatrix}$$

$$\text{maximize}\ p=5x+4y \quad \text{s.t.}\ 3x+5y\leq 78,\ 4x+y\leq 36,\ x,y\succeq 0$$

$\Updownarrow$

$$\underset{x\,y\,s_1\,s_2}{\text{Maximize}}\ p=5x+4y \quad 3x+5y+s_1=78,\ 4x+y+s_2=36,\ x,y,s_1,s_2\geq 0$$

$$\begin{array}{cccccc|c} x&y&s_1&s_2&p&&\\ 3&5&1&0&0&78&26\\ 4&1&0&1&0&36&9\\ -5&-4&0&0&1&0& \end{array}$$

$\Downarrow$

$$\begin{bmatrix}3&5&1&0&0&78\\1&1/4&0&1/4&0&9\\-5&-4&0&0&1&0\end{bmatrix}$$

$\Downarrow$

$$\begin{array}{cccccc|c} 0&17/4&1&-3/4&0&51&12\\ 1&1/4&0&1/4&0&9&36\\ 0&-11/4&0&5/4&1&45& \end{array}$$

$\Downarrow$

$$\begin{bmatrix}0&1&4/17&-3/17&0&12\\1&1/4&0&1/4&0&9\\0&-11/4&0&5/4&1&45\end{bmatrix}$$

$\Downarrow$

$$\begin{bmatrix}0&1&4/17&-3/17&0&12\\1&0&-\frac{1}{17}&\frac{5}{17}&0&6\\0&0&\frac{11}{17}&\frac{13}{17}&1&78\end{bmatrix}$$

$x=6$
$y=12$
$P=78$

- **Integer/Mixed-Integer Linear Programming**

ILP/MILP

$$\underset{x}{\text{minimize}}\ c^Tx \quad \text{s.t.}\ Ax=b,\ x\succeq 0,\ x\in\{x\mid x_i\in\mathbb{Z},\ i=1\sim l\}$$

- Solution – Branch & Bound
  - Branch:
    partition D into subsets
    (check objective value is subsets)
  - Bound:
    acquire a bound w.o./ integer constraints

e.g.

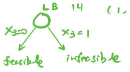
$$\underset{x}{\text{maximize}}\ 9x_1+5x_2+6x_3 \quad \text{s.t.}\ 6x_1+3x_2+5x_3\leq 10,\ -x_1+x_3\leq 0,\ x_i\in\{0,1\}\ i=1,2,3$$

$\Updownarrow$

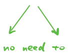
$$\underset{x}{\text{maximize}}\ 9x_1+5x_2+6x_3 \quad \text{s.t.}\ 6x_1+3x_2+5x_3\leq 10,\ -x_1+x_3\leq 0,\ x_i\leq 1\ i=1,2,3,\ x_i\geq 0\ i=1,2,3,\ x_i\in\mathbb{Z}\ i=1,2,3$$

△ Bound:
- w.o./ $\mathbb{Z}$ constraints:
  $x_1=1$
  $x_2=1$
  $x_3=0.2$
  $P=15.2$ (UB)
- Rounded down solution
  $x_1=1$
  $x_2=1$
  $x_3=0$
  $p=14$ (LB)

△ Branch
- UB 15.2 (1,1,0.2)
  LB 14 (1,1,0)



UB 14 (1,1,0)
LB 14 (1,1,0)



no need to check hv
UB=LB

- **Solutions to Optimization Problems.**
  - recall Duality

$$\underset{x}{\text{minimize}}\ f_0(x) \quad \text{s.t.}\ f_i(x)\leq 0,\ h_i(x)=0$$

△ Lagrangian:

$$\mathcal{L}(x,\lambda,\nu)=f_0(x)+\sum_{i=1}^{m}\lambda_if_i(x)+\sum_{i=1}^{p}\nu_ih_i(x)$$

△ Lagrangian Dual function:

let

$$g(\lambda,\nu)=\inf_{x\in D}\{\mathcal{L}(x,\lambda,\nu)\}$$

$$\underset{\lambda,\nu}{\text{maximize}}\ g(\lambda,\nu) \quad \text{s.t.}\ \lambda\succeq 0$$

△ let $\lambda^*,\nu^*,\ d^*=g(\lambda^*,\nu^*)$ be optimal

$p^*\geq d^*$

△ Strong duality

Slater's condition

$$\exists x\in \text{relint}\ D \quad \text{s.t.}\ f_i(x)<h_i,\ h_i(x)=0$$

△ complementary slackness

$$\begin{aligned}f_0(x^*)&=g(\lambda^*,\nu^*)\\&=\inf_x\ f_0(x)+\sum_{i=1}^m\lambda_i^*f_i(x)+\sum_{i=1}^p\nu_i^*h_i(x)\\&\leq f_0(x^*)+\sum_{i=1}^m\lambda_i^*f_i(x^*)+\sum_{i=1}^p\nu_i^*h_i(x^*)\\&\leq f_0(x^*)\end{aligned}$$

$$\lambda_i^*f_i(x^*)=0$$

$$\Rightarrow\begin{cases}f_i(x^*)<0\rightarrow\lambda_i^*=0\\ \lambda_i^*>0\rightarrow f_i(x^*)=0\end{cases}$$

△ KKT optimality conditions

$$\begin{cases}f_i(x^*)\leq 0\\ h_i(x^*)=0\\ \lambda^*\succeq 0\\ \lambda_i^*f_i(x^*)=0\\ \nabla f_0(x^*)+\sum_{i=1}^m\lambda_i\nabla f_i(x^*)+\sum_{i=1}^p\nu_i\nabla h_i(x^*)=0\end{cases}$$

this will be used later

- **Descent Method**

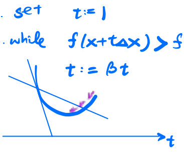
1. given $x^{(0)}\in\text{dom}\,f$
2. repeat
   1) descent direction Δx
   2) stepsize $t\geq 0$
   3) $x^{k+1}=x^k+t^k\Delta x^k$
3. Stop
   $\|\Delta f(x)\|_2\leq\eta$

- **Line Search**
  - Exact Line Search

$$t=\underset{s\geq 0}{\text{argmin}}\ f(x+s\Delta x)$$

  - Backtracking Line Search

1. given Δx^k, $f(x^k)$, $\alpha\in(0,0.5)$, $\beta\in(0,1)$
2. set $t:=1$
3. while $f(x+t\Delta x)>f(x)+\alpha t\nabla f(x)^T\Delta x$
   $t:=\beta t$



- **Primal – dual Interior-point method**

$$\text{minimize}\ tf_0(x)+\phi(x) \quad \text{s.t.}\ Ax=b$$

KKT:

$$\begin{cases}Ax^*(t)=b\\ t\nabla f_0(x^*(t))+\nabla\phi(x^*(t))+A^T\hat{\nu}=0\end{cases}$$

$$\Rightarrow\begin{cases}Ax^*(t)=b\\ t\nabla f_0(x^*(t))+\sum_{i=1}^m\frac{1}{-f_i(x^*(t))}\nabla f_i(x^*(t))+A^T\hat{\nu}=0\end{cases}$$

$$\Rightarrow\begin{cases}Ax=b\\ \nabla f_0(x)+\sum_{i=1}^m-\frac{1}{f_i(x)t}\nabla f_i(x)+A^T\nu=0\end{cases}\qquad \frac{\hat{\nu}}{t}=\nu$$

$$\Rightarrow\begin{cases}\nabla f_0(x)+\sum_{i=1}^m\lambda_i\nabla f_i(x)+A^T\nu=0\\ -\lambda_if_i(x)=1/t\\ Ax=b\end{cases}$$

$$\Rightarrow r_t(x,\lambda,\nu)=\begin{bmatrix}\nabla f_0(x)+Df(x)^T\lambda+A^T\nu\\-\text{diag}(\lambda)f(x)-1/t\,\mathbf{1}\\Ax-b\end{bmatrix}\begin{matrix}\leftarrow r_{dual}\\ \leftarrow r_{cent}\\ \leftarrow r_{primal}\end{matrix}$$

$$\Rightarrow r_t(x+\Delta x,\lambda+\Delta\lambda,\nu+\Delta\nu)=\begin{bmatrix}\nabla^2f_0(x)+\sum_{i=1}^m\lambda_i\nabla^2f_i(x)&Df(x)^T&A^T\\-\text{diag}(\lambda)Df(x)&-\text{diag}(f(x))&0\\A&0&0\end{bmatrix}\begin{bmatrix}\Delta x\\\Delta\lambda\\\Delta\nu\end{bmatrix}=\begin{bmatrix}-r_{dual}\\-r_{cent}\\-r_{pri}\end{bmatrix}$$

- **Unconstrained problem**
  - Gradient descent

1. given $x_0\in\text{dom}\,f$
2. repeat
   - $\Delta x^k=-\nabla f(x)$
   - t: from line search
   - $x^{k+1}=x^k+t\Delta x^k$
3. until
   $\|\nabla f(x)\|\leq\eta$

  - Newton's method

1. given: $x^0\in\text{dom}\,f$
2. repeat:
   $\Delta x^k=-\nabla^2f(x^k)^{-1}\nabla f(x^k)$
   $\lambda^{k^2}=\nabla f(x^k)\nabla^2f(x^k)\nabla f(x^k)$
   t: from line search
   $x^{k+1}=x^k+t\Delta x$
3. until
   $\lambda^{k^2}/2\leq\epsilon$

on Newton's Method

$$f(x)$$
$$\Rightarrow\hat{f}(x+v)=f(x)+\nabla f(x)^Tv+\tfrac{1}{2}v^T\nabla^2f(x)v$$
$$\Rightarrow\nabla\hat{f}(x+v)=0$$
$$\Rightarrow\nabla f(x)+\nabla^2f(x)v=0$$
$$\Rightarrow v=-\nabla^2f(x)^{-1}\nabla f(x)$$

- **Equality-constrained Problem**

$$\text{minimize}\ f(x) \quad \text{s.t.}\ Ax=b$$

recall
Slater's condition
$\exists x\in\text{relint}\,D$ s.t. $Ax=b$
KKT optimality condition

$$f_i(x^*)\leq 0$$
$$h_i(x^*)=0$$
$$\lambda^*\succeq 0$$
$$\lambda_i^*f_i(x^*)=0$$
$$\nabla f_0(x^*)+\sum_{i=1}^m\lambda_i^*\nabla f_i(x^*)+\sum_{i=1}^p\nu_i^*\nabla h_i(x^*)=0$$

△ Here @ optimal $\tilde{x},\tilde{\nu}$

$$\begin{cases}A\tilde{x}=b\\ \nabla f(\tilde{x})+A^T\tilde{\nu}=0\end{cases}$$

- Newton's Method

1. @ x
   addition to x: $x+v$
   $\therefore x+v\in\text{dom}\,f$
   $A(x+v)=b$
   $\therefore Av=0$ — ①
2. $\hat{f}(x+v)$
   $=f(x)+\nabla f(x)v+\frac{1}{2}v^T\nabla^2f(x)v$

$$\therefore\ \text{minimize}\ f(x)+\nabla f(x)v+\tfrac{1}{2}v^T\nabla^2f(x)v \quad \text{s.t.}\ Av=0 \ —②$$

③ $\tilde{v}$ is optimal of ②

$\therefore$ $\exists w$ s.t. ⤳ Lagrange multiplier of ②

$$\nabla f(x)+\nabla^2f(x)\tilde{v}+A^Tw=0$$

and recall $A\tilde{v}=0$

→ from KKT $\mathcal{L}(v,w)=\hat{f}(v)+w(Av)$

$$\therefore\ A\tilde{v}=0$$
$$\nabla f(x)+\nabla^2f(x)\tilde{v}+A^Tw=0$$

$$\Rightarrow\begin{bmatrix}\nabla^2f(x)&A^T\\A&0\end{bmatrix}\begin{bmatrix}\tilde{v}\\w\end{bmatrix}=\begin{bmatrix}-\nabla f(x)\\0\end{bmatrix}$$

$$\therefore\ \tilde{v}=\Delta x_{nt}.$$

- **Inequality-Constrained Problem**
  - Barrier Method

$$\text{minimize}\ f_0(x) \quad \text{s.t.}\ f_i(x)\leq 0,\ Ax=b$$

$\Updownarrow$

$$\text{minimize}\ tf_0(x)+\phi(x) \quad \text{s.t.}\ Ax=b \qquad \phi(x)=-\sum_{i=1}^m\log(-f_i(x))$$

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1. given $x^0\in\text{dom}\,f$
   $t^0:=t^0>0$
   $\mu>1$
   $\epsilon>0$
2. repeat
   a. solve
      $\text{minimize}\ tf_0(x)+\phi(x)$ s.t. $Ax=b$
      (get x^*, w t^k)
   b. $x:=x^*(t)$
   c. until $m/t<\epsilon$
   d. $t:=\mu t$