

- Review - high order linear ODEs
- characteristic eqⁿ
- matrix system of 1st order ODE $\dot{y} = Ay$

- Example

- Special case $\dot{y} = Dy$ - D diagonal

- Derive eigenvalue equation to diagonalize any system

Higher Order ODE (linear)

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

try $x(t) = e^{\lambda t} \rightarrow x^{(n)} = \lambda^n x$

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0) \underline{\underline{x(t)}} = 0$$

Characteristic polynomial

↓
in German
↓
eigen!

In general, n solutions: $\lambda_1, \lambda_2, \dots, \lambda_n$

★ $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t} \rightarrow$ general solution

use $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix}$ to solve $\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$

(cont'd)

$$x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \dots + a_3 \ddot{x} + a_2 \dot{x} + a_1 x + a_0 x = 0$$

$$\begin{aligned} X_1 &= x \\ X_2 &= \dot{x} \\ X_3 &= \ddot{x} \\ &\vdots \\ X_n &= x^{(n-1)} \end{aligned}$$

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= X_3 \\ \dot{X}_3 &= X_4 \\ &\vdots \\ \dot{X}_n &= -a_0 X_1 - a_1 X_2 - a_2 X_3 - \dots - a_{n-1} X_n \end{aligned}$$

$$\frac{d}{dt} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix}$$

$$\dot{y} = A y$$

General

$$0 = (\lambda^n) (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_{n-1} \lambda^{n-1} + \lambda^n)$$

eg.

$$\ddot{x} + 3\dot{x} + 2x = 0$$

⇓

$$\left. \begin{aligned} \dot{x} &= v \\ \dot{v} &= -2x - 3v \\ &= -2x - 3v \end{aligned} \right\} \frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_A \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\dot{y} = A y$$

eigenvalues of A are roots of characteristic polynomials

$$\det(A - \lambda I) = 0$$

$$\det \left(\begin{bmatrix} A & - \lambda I \\ \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} & - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \end{bmatrix} \right) = \det \left(\begin{bmatrix} -\lambda & 1 \\ -2 & -\lambda-3 \end{bmatrix} \right) = \lambda^2 + 3\lambda + 2 = 0$$

no want to solve general systems $\dot{X} = AX$

Case 1: Uncoupled Dynamics (only depends themselves)

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{e.g. } \begin{matrix} \text{you} \\ \text{animal} \end{matrix} \text{ reproducing}$$

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 & x_1(t) &= e^{\lambda_1 t} x_1(0) \\ \dot{x}_2 &= \lambda_2 x_2 & \Rightarrow x_2(t) &= e^{\lambda_2 t} x_2(0) \\ & \vdots & & \vdots \\ \dot{x}_n &= \lambda_n x_n & x_n(t) &= e^{\lambda_n t} x_n(0) \end{aligned}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} \Rightarrow \boxed{X(t) = e^{At} X(0)}$$

we need a coord transformation

$$X = Tz$$

so that $\dot{z} = \underline{D} z$
Diagonal

$$\begin{aligned} Tz &= X \\ \Rightarrow T\dot{z} &= \dot{X} = AX = ATz \\ \Rightarrow T\dot{z} &= ATz \\ \Rightarrow \dot{z} &= \underbrace{T^{-1}AT}_{D \text{ (we hope)}} z \end{aligned}$$

$$\begin{aligned} T^{-1}AT &= D \\ TT^{-1}AT &= TD \end{aligned}$$

* $\boxed{AT = TD}$
 eigenvalue equation

eigenvectors of A
 \hookrightarrow define a new coordinate system
 \hookrightarrow where on that vector, everything will be decoupled
 eigenvector

$$\begin{aligned} \begin{bmatrix} A \\ A \\ \vdots \\ A \end{bmatrix} \begin{bmatrix} | & | & \dots & | \\ t_1 & t_2 & \dots & t_n \\ | & | & \dots & | \end{bmatrix} &= \begin{bmatrix} | & | & \dots & | \\ t_1 & t_2 & \dots & t_n \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_n \end{bmatrix} \\ &= T^{-1}AT = D \end{aligned}$$

$$\begin{aligned} At_1 &= \lambda_1 t_1 \\ At_2 &= \lambda_2 t_2 \\ & \vdots \\ At_n &= \lambda_n t_n \end{aligned}$$

$$\Rightarrow [T, D] = \text{eig}(A);$$

$$\begin{aligned} D &= T^{-1}AT \\ A &= TDT^{-1} \\ A^2 &= TD^2T^{-1} \end{aligned}$$

$$A^3 = TD^3T^{-1}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots = Te^{Dt}$$