ME564 L5

1. 2nd order ODEs
2. matlab
3. higher order ODES

Next week: $\dot{x} = Ax$; $x(0)$
eigenvalues & e-vecs.

Example

$\ddot{x} + 3\dot{x} + 2x = 0$ I.C. $x(0) = 2$ $\dot{x}(0) = -3$

$$\left.\begin{aligned} x(t) &= e^{\lambda t} \\ \dot{x}(t) &= \lambda e^{\lambda t} \\ \ddot{x}(t) &= \lambda^2 e^{\lambda t} \end{aligned}\right\} (\lambda^2 + 3\lambda + 2) e^{\lambda t} = 0$$

$\lambda^2 + 3\lambda + 2 = 0$ (characteristic equation)

$(\lambda+1)(\lambda+2) = 0$

$\lambda = -1, -2$

$\therefore$ $x(t) = k_1 e^{-t} + k_2 e^{-2t}$

$$\left.\begin{aligned} x(0) &= k_1 + k_2 \\ \dot{x}(0) &= -k_1 - 2k_2 \end{aligned}\right\} \Rightarrow k_1 = k_2 = 1$$

$$\boxed{\therefore\ x(t) = e^{-t} + e^{-2t}}$$

Example

$\ddot{x} + 3\dot{x} + 2x = 0$

$\Downarrow$ suspend variables (could do it for any differential equation

$$\begin{cases} \dot{x} = v \\ \dot{v} = -2x - 3\dot{x} \end{cases}$$

$\dot{v} = -2x - 3v$

$$\left\{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2x - 3v \end{aligned}\right\} \Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$ ☆ useful for matlab

$\dot{y} = Ay$ $\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

ple

e.g. — damping

$\ddot{x} - 3\dot{x} + 2x = 0 \quad x(0) = 2 \quad \dot{x}(0) = +3$

$x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t} \quad [\lambda^2 - 3\lambda + 2] e^{\lambda t} = 0$

$\ddot{x}(t) = \lambda^2 e^{\lambda t} \quad \lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0 \Rightarrow \lambda = +1, \ \lambda = +2$

$x(t) = k_1 e^{t} + k_2 e^{2t} \qquad k_1 = k_2 = 1$

$x(t) = e^{t} + e^{2t}$

λ can give me a sense of how the system should behave

$\lambda \in$ complex no. $\rightarrow$ oscillating

$\lambda < 0 \in$ converge

$\lambda > 0 \in$ diverge

Example

$\ddot{x} + \dot{x} - 2x = 0 \qquad x(0) = 3 \quad \dot{x}(0) = 0$

$x = e^{\lambda t}$

$\dot{x} = \lambda e^{\lambda t} \qquad [\lambda^2 + \lambda - 2] e^{\lambda t} = 0$

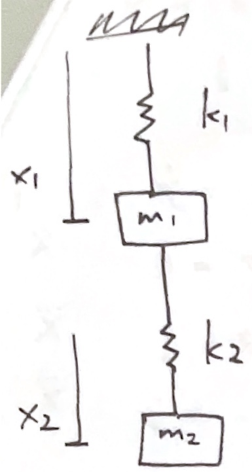
$\ddot{x} = \lambda^2 e^{\lambda t} \qquad (\lambda^2 + \lambda - 2) = 0$

$(\lambda + 2)(\lambda - 1) = 0 \Rightarrow \lambda = 1, \lambda = -2$

$x(t) = k_1 e^{t} + k_2 e^{-2t}$

$= \underbrace{e^{t}}_{\to \infty} + \underbrace{e^{-2t}}_{\to 0}$

system unstable (but can still be stable if I.C. is set properly)



$$\begin{cases} m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \end{cases}$$

- coupling
- second order
- linear

$\downarrow$

system of 4 first order equation

$\dot{x}_1 = V_1$
$\dot{V}_1 = \dots$
$\dot{x}_2 = V_2$
$\dot{V}_2 = \dots$

or

or

single 4^{th} order equation

solve ① for $x_2 = f(x_1)$

take 2^{nd} derivative $\ddot{x}_2 = \frac{d}{dt^2} f(x_1)$

plug in to ②

4^{th} order

$$\ddddot{x} + 5\dddot{x} + 2\ddot{x} + \dot{x} + 7x = 0$$

$x = e^{\lambda t}$
$\dot{x} = \lambda e^{\lambda t}$
$\ddot{x} = \lambda^2 e^{\lambda t}$
$\dddot{x} = \lambda^3 e^{\lambda t}$
$\ddddot{x} = \lambda^4 e^{\lambda t}$
$\vdots$

$(\lambda^4 + 5\lambda^3 + 2\lambda^2 + \lambda + 7) e^{\lambda t} = 0$

$\lambda^4 + 5\lambda^3 + 2\lambda^2 + \lambda + 7 = 0$ characteristic equation.

4 sol. : $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + k_3 e^{\lambda_3 t} + k_4 e^{\lambda_4 t}$

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① second ODEs

② matlab

③ higher order ODES

Example

$\ddot{x} + 3\dot{x} + 2x = 0$

$x(0) = 2 \; ; \; \dot{x}(0) = -3$

for linear cases: $\rightarrow e^{\lambda t}$

$\left.\begin{array}{l} x(t) = e^{\lambda t} \\ \dot{x}(t) = \lambda e^{\lambda t} \\ \ddot{x}(t) = \lambda^2 e^{\lambda t} \end{array}\right\} \quad [\lambda^2 + 3\lambda + 2]\, \underbrace{e^{\lambda t}}_{>0} = 0$

$\lambda^2 + 3\lambda + 2 = 0 \longrightarrow$ characteristic equation

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -1, -2$

$x(t) = k_1 e^{-t} + k_2 e^{-2t}$

$\left.\begin{array}{l} x(0) = k_1 + k_2 \\ \dot{x}(0) = -k_1 - 2k_2 \end{array}\right\} \Rightarrow \quad k_1 = k_2 = 1$

$x(t) = e^{-t} + e^{-2t}$ ✲

```
t = 0:0.02:10;
x = exp(-t) + exp(-2*t);
plot(t, x)
```

$\ddot{x} + 3\dot{x} + 2x = 0$ 1 second ODE

⇓ suspend variables

$\dot{x} = v$

$\dot{v} = -2x - 3\dot{x}$

$= -2x - 3v$

$\left.\begin{array}{l} \dot{x} = v \\ \dot{v} = -2x - 3v \end{array}\right\}$ 2 first order equations $\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$ $y_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\dot{y} = Ay$ ↳ useful for MatLab

```
y0 = [2; -3];
A = [0 1; -2 -3]
[t, y] = ode45(@(t, y) A*y, t, y0)
```

$\ddot{x} + \dot{x} - 2x = 0$

$x(0) = 3$

$\dot{x}(0) = 0$

$x = e^{\lambda t}$

$\dot{x} = \lambda e^{\lambda t}$ $\quad [\lambda^2 + \lambda - 2]\, e^{\lambda t} = 0$

$\ddot{x} = \lambda^2 e^{\lambda t}$ $\quad \lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda = -2, 1$

$x(t) = k_1 e^{t} + k_2 e^{-2t}$ $\qquad k_1 + k_2 = 3$

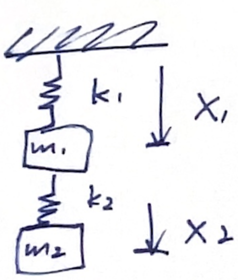
$= \underbrace{2e^{t}}_{\to\infty} + \underbrace{e^{-2t}}_{\to 0}$ $\qquad k_1 - 2k_2 = 0$

$k_1 = 2$

$k_2 = 1$

∴ unstable system

unstable term always dominant!



$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0$ — 1)

$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0$ — 2)

① system of 4 1st order equations

$\dot{x}_1 = v_1$

$\dot{v}_1 = \dots$

$\dot{x}_2 = v_2$

$\dot{v}_2 = \dots$

② or single 4th order equations

solve 1) for $x_2 = f(x_1)$

take 2 derivatives $\ddot{x}_2 = \frac{d^2}{dt^2} f(x_1)$

plug in to 2)

$$\ddddot{x} + 5\dddot{x} + 2\ddot{x} + \dot{x} + 7x = 0$$

$x = e^{\lambda t}$

$\dot{x} = \lambda e^{\lambda t}$ $\qquad [\lambda^4 + 5\lambda^3 + 2\lambda^2 + \lambda + 7] e^{\lambda t} = 0$

$\ddot{x} = \lambda^2 e^{\lambda t}$

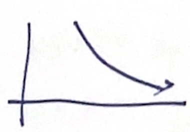
$\dddot{x} = \lambda^3 e^{\lambda t}$ $\qquad \lambda^4 + 5\lambda^3 + 2\lambda^2 + \lambda + 7 = 0 \rightarrow$ characteristic eqn.

$\ddddot{x} = \lambda^4 e^{\lambda t}$ $\qquad$ 4 solns $\quad \lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$

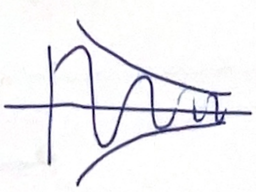
$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + k_3 e^{\lambda_3 t} + k_4 e^{\lambda_4 t}$$

$\lambda > 0$

$\lambda < 0$



$\lambda \rightarrow$ complex no.



$$\ddddot{x} + 5\dddot{x} + 2\ddot{x} + \dot{x} + 7x = 0$$

$\downarrow$ introduce dummy variables

$x \quad x$

$\dot{x} \quad \dot{x} = y$

$\ddot{x} \quad \dot{y} = z$

$\dddot{x} \quad \dot{z} = a$

$\ddddot{x} \quad \dot{a} = -7x - \dot{x} - 2\ddot{x} - 5\dddot{x}$

$= -7x - y - 2z - 5a$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & -1 & -2 & -5 \end{bmatrix}}_{\lambda \text{ are eigs}(A)} \begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix}$$