ME564 L27

① - Start w/ PDE
(Laplace's Eqⁿ)

② - Get a vector field $\vec{v}(x,y)$

③ Use $\vec{v}$ as RHS for particle trajectory
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix}$$

2D potential flow $\quad \varphi = \varphi(x,y)$

$\vec{v} = \nabla\varphi$ where $\left.\begin{matrix} \nabla\cdot\vec{v} = 0 \\ \nabla\times\vec{v} = 0 \end{matrix}\right\}$ iff $\underline{\nabla^2\varphi = 0}$

φ is a very special function (potential)

Another special function ψ (stream function)

$\underline{\nabla^2\psi = 0}$

$$\Phi(z) = \varphi(x,y) + i\,\psi(x,y)$$

$z = x+iy$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \partial\varphi/\partial x \\ \partial\varphi/\partial y \end{bmatrix} = \nabla\varphi = \begin{bmatrix} \partial\psi/\partial y \\ -\partial\psi/\partial x \end{bmatrix}$$

(phi) (psi)

$\Phi(z) = z^n$
building block

(one function could satisfy laplace (when $n\in\mathbb{N}$) but why?)

$$z^2 = \underbrace{x^2-y^2}_{\varphi} + 2i\underbrace{xy}_{\psi}$$

$\varphi(x,y) = x^2-y^2$

$\psi(x,y) = 2xy$

1. φ & ψ satisfy $\nabla^2\varphi = 0$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 2-2 = 0$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0+0 = 0$$

Laplace.

KOKUYO LOOSE-LEAF ノ-816B 8 mm ruled × 41 lines

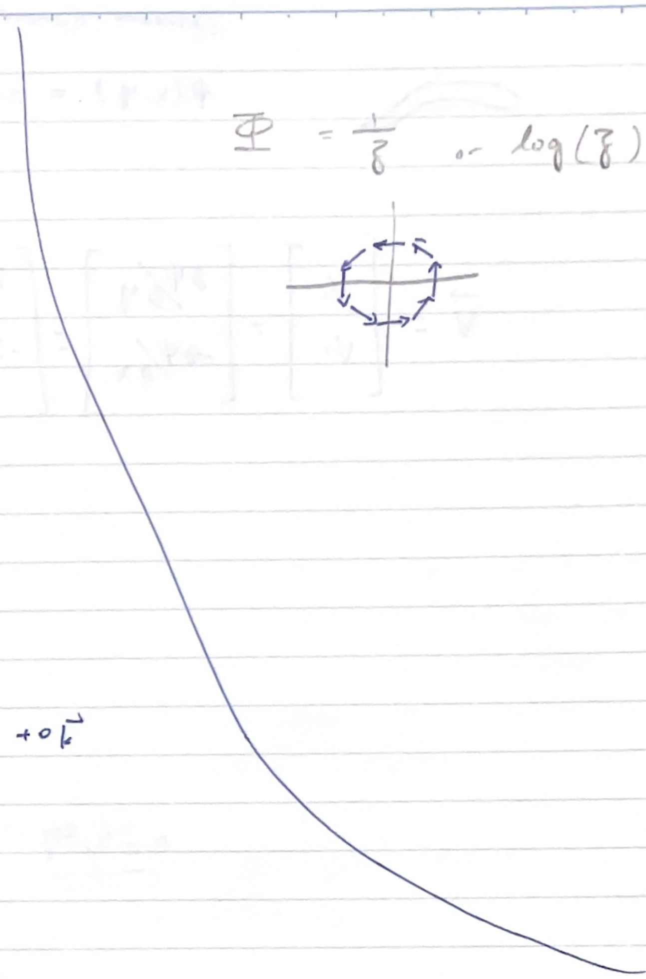
(cont'd)

compute $\vec{V}$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

$$\begin{array}{cc} \partial\varphi/\partial x & \partial\psi/\partial y \\ \partial\varphi/\partial y & -\partial\psi/\partial x \end{array}$$

$\Phi = \frac{1}{z}$ or $\log(z)$



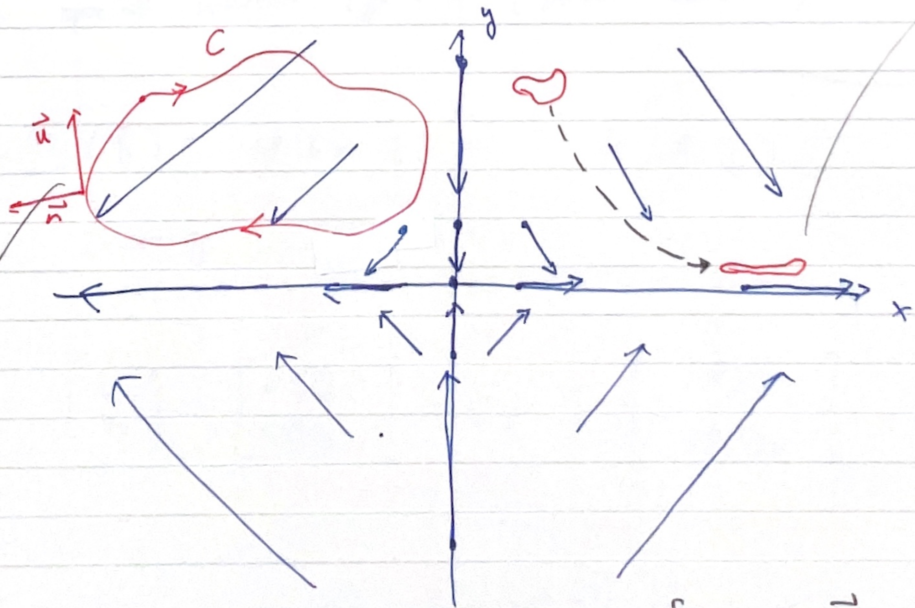
3. $\vec{V}$ is incompressible & irrotational

$$\nabla \cdot \vec{V} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \cdot \begin{bmatrix} 2x \\ -2y \end{bmatrix} = 0$$

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & -2y & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

plot

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$



with t ↑
it maneuvers,
streched in x
compressed in y
yet volume doesn't change
→ incompressible, irrotational.

$\dot{x} = V_1(x, y)$

$\dot{y} = V_2(x, y)$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

ODE
induced by
my velocity field!

true for any $\vec{V} = \nabla\varphi$

Circulation: $\oint_C \vec{V} \cdot \vec{u} \, ds = \iint_{\text{inside } C} \nabla \times \vec{V} \, dA = 0$ Stokes.

Flux: $\oint_C \vec{V} \cdot \vec{n} \, ds = \iint_{\text{inside } C} \nabla \cdot \vec{V} \, dA = 0$

true for $\vec{V} = \nabla\varphi$ & $\nabla^2\varphi = 0$

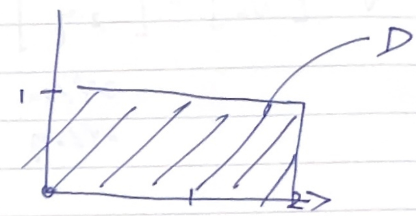
No.

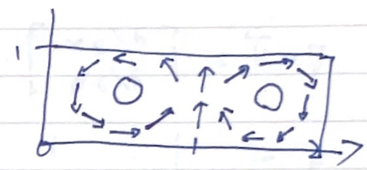
Date

(stream function)

$$\psi(x,y) = \sin(\pi x)\sin(\pi y) \quad \text{on } D = [0,2]\times[0,1]$$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial\psi/\partial y \\ -\partial\psi/\partial x \end{bmatrix} = \begin{bmatrix} \pi\sin(\pi x)\cos(\pi y) \\ -\pi\cos(\pi x)\sin(\pi y) \end{bmatrix}$$





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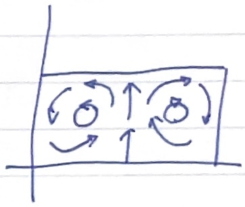
Date

ME564 L28

① Double Gyre Flow $\longrightarrow$ time-invariant

$$\psi(x,y) = \sin(\pi x)\sin(\pi y) \quad \text{on} \quad D = [0,2]\times[0,1]$$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial\psi/\partial y \\ -\partial\psi/\partial x \end{bmatrix} = \begin{bmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi\cos(\pi x)\sin(\pi y) \end{bmatrix}$$



$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

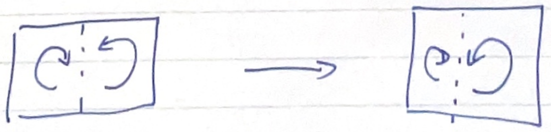
$$\left.\begin{aligned} \dot{x} &= V_1(x,y) \\ \dot{y} &= V_2(x,y) \end{aligned}\right\} \quad \text{ODE45, pplane.}$$

② Double Gyre Flow $\rightarrow$ time-variant

$$\psi(x,y) = \sin(\pi x)\sin(\pi y)$$

$$\psi(x,y) = \sin(\pi f(x,t)\, x)\sin(\pi y) \qquad f(x,t) = ax^2 + bx \qquad a(t) = \epsilon\sin(\omega t)$$

$$b = 1 - 2a$$



(varies in space & time).

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial\psi/\partial y \\ -\partial\psi/\partial x \end{bmatrix}$$