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MES64 L23

Gauss's Divergence Theorem

Overview of Topics

① Double & Triple Integrals

② Gauss's Thm $\nabla \cdot$.

* Example

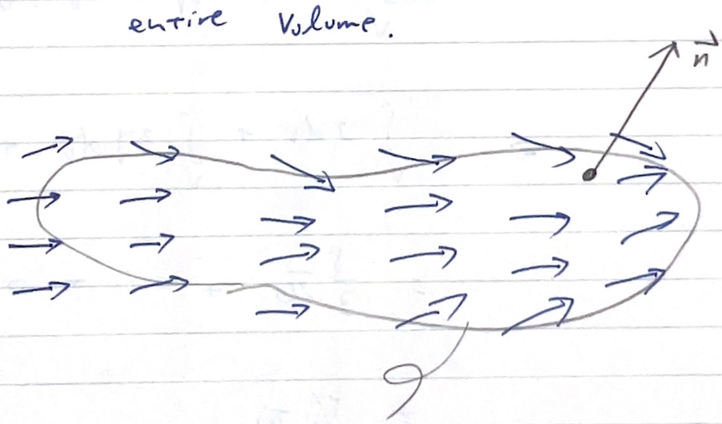
* Continuity Eqn.

Gauss's Divergence Theorem

→ In words : The flux of a vector field out of a close surface

how many stuff is being carried in or out of my volume.

= the integral of the $\nabla \cdot$ (divergence) of the vector field over the entire volume.



∂V (boundary of my volume).

in math

$$\underbrace{\iint_{\partial V} \vec{F} \cdot \vec{n} \, dS}_{\text{flux of mass through } \partial V} = \iiint_{V} \nabla \cdot \vec{F} \, dV$$

flux of mass through ∂V

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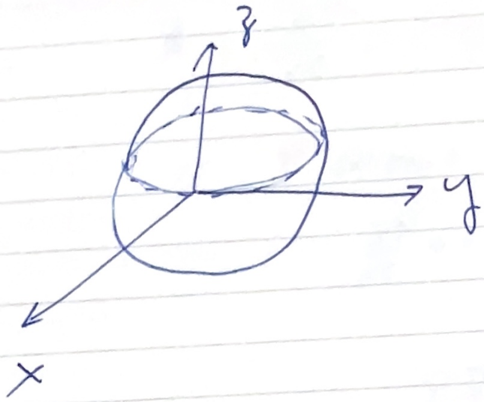


$\vec{F} = 2x\vec{i} + y^2\vec{j} + z^2\vec{k}$

$S = \{x^2+y^2+z^2=1$

compute flux through S

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

$\Rightarrow \quad \nabla\cdot\vec{F} = 2+2y+2z$

$\Rightarrow \quad \iiint_V (2+2y+2z)\,dV$

$\Rightarrow \quad \int_V 2\,dV + \int_V 2y\,dV + \int_V 2z\,dV.$

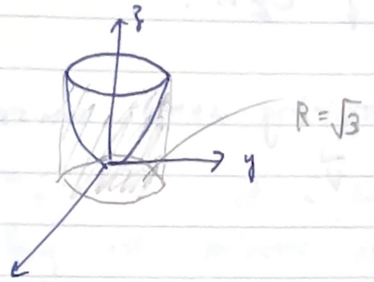
$= \frac{8}{3}\pi + 0 + 0$

$= 8/3\,\pi$

ex

$f(x,y) = x^2+y^2$

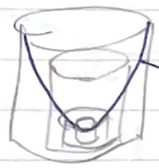
$= r^2$



find volume under $f(x,y)$

for $0 \le x^2+y^2 \le 3$

$0 \le r^2 \le 3$



$$V = \iiint dV = \int_{r=0}^{\sqrt{3}} \int_{\theta=0}^{2\pi} \int_{z=0}^{f(r)} dz\, d\theta\, dr.$$

$$= \int_{r=0}^{\sqrt{3}} \int_{0}^{2\pi} r^2\, d\theta\, dr$$

$$= \int_0^{\sqrt{3}} \overbrace{2\pi r^2}^{2\pi h \Delta r} dr$$

$$= \left[2\pi \frac{r^3}{3} \right]_0^{\sqrt{3}}$$

$$= 2\pi\sqrt{3}$$

$$\vec{F} = (3x - 2xy)\vec{i} - y\vec{j} + 2yz\vec{k}$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV = \iiint_V 2\, dV = 2\cdot 4\pi\sqrt{3}$$

$$\nabla\cdot\vec{F} = 3 - 2y - 1 + 2y = 2$$

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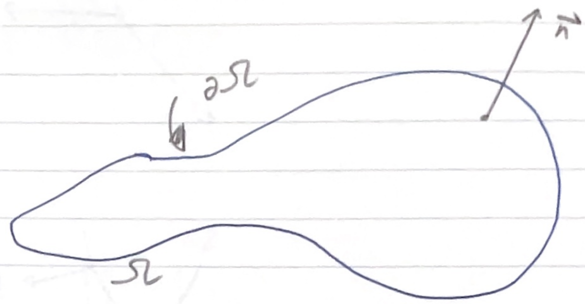
Conservation of Mass & Continuity Eqn.

→ consider a volume Ω containing some mass (fluid)
we also hv a velocity field $\vec{v}$



let ρ be the density.

Any change in mass in Ω must correspond to mass being carried in/out by $\vec{v}$

$$\frac{d}{dt}\iiint_{\Omega} \rho \, d\Omega = -\iint_{\partial\Omega} \rho \vec{v} \cdot \vec{n} \, dS$$

rate of change Δ in mass.

$$= -\iiint_{\Omega} \rho \, \nabla \cdot \vec{v} \, d\Omega$$

→

$$\iiint_{\Omega} \left[\frac{d}{dt} \rho + \nabla \cdot (\rho \cdot \vec{v}) \right] d\Omega = 0$$

Integral form of mass conservation!

for all Ω!! $\Rightarrow$ $\boxed{\frac{d}{dt} \rho + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{everywhere!}}$

linear PDE

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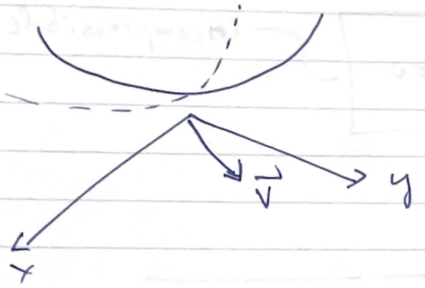
△ directional derivative

the gradient ∇f of a function f can be used to compute the directional derivative of f in some vector direction $\vec{V}$



$$D_{\vec{V}} f = \frac{(\nabla f)\cdot \vec{V}}{\|\vec{V}\|} \qquad \nabla f = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$$

if $\vec{V} = \vec{i}$: $\frac{\partial f}{\partial x}$

if $\vec{V} = \vec{j}$: $\frac{\partial f}{\partial y}$

if $\vec{V} = \vec{k}$: $\frac{\partial f}{\partial z}$

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\vec{V}) = 0$$

$$\nabla\cdot(\rho\vec{V}) = \nabla\cdot\begin{bmatrix}\rho V_1 \\ \rho V_2 \\ \rho V_3\end{bmatrix} = \overset{\frac{\partial}{\partial x}}{(\rho_x V_1 + \rho V_{1x})} + \overset{\frac{\partial}{\partial y}}{(\rho_y V_2 + \rho V_{2y})} + \overset{\frac{\partial}{\partial z}}{(\rho_z V_3 + \rho V_{3z})}$$

$$= \rho\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + (\rho_x V_1 + \rho_y V_2 + \rho_z V_3)$$

$$= \rho\,\nabla\cdot\vec{V} + \underbrace{(\nabla\rho)\cdot\vec{V}}_{\|\vec{V}\|\cdot D_{\vec{V}}\rho}$$

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★ $$\frac{\partial \rho}{\partial t} + \rho \nabla\cdot\vec{v} + (\nabla\rho)\cdot\vec{v} = 0$$

if ρ constant everywhere

$$\frac{\partial \rho}{\partial t} = 0 \quad \& \quad \nabla\rho = \vec{0} \quad \ldots \quad D_{\vec{v}}\,\rho = 0 \text{ for all } \vec{v}$$

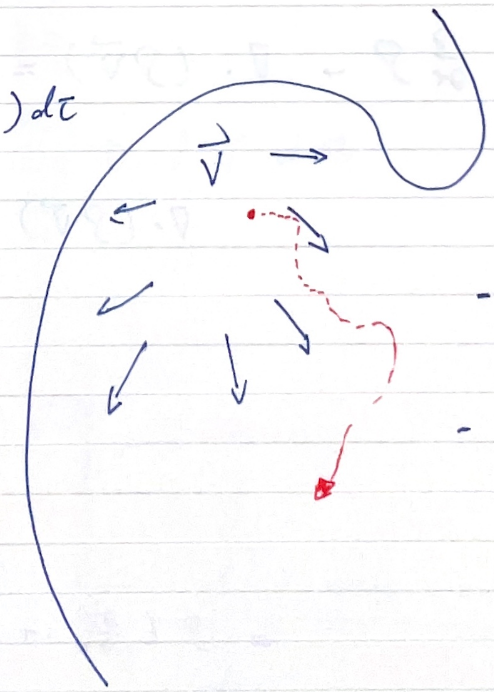
$$\rho\,\nabla\cdot\vec{v} = 0 \Rightarrow \boxed{\nabla\cdot\vec{v} = 0}$$ incompressible

Vector field $\vec{v}$

① $\vec{v}$ is often the sol^{n} of a PDE. (Navier-Stokes)

② $\dot{x} = \vec{v}(x,t)$

$$x(t) = x_0 + \int_0^t \vec{v}(x(\tau),\tau)\,d\tau$$



- solve $\vec{v}$ from PDE
- ODE then gives me the phenomenon along some variable (e.g. t)