

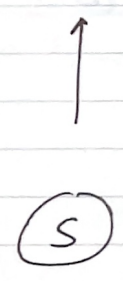
MES/4 L17

Numerical \rightarrow we value : stability
accuracy

$\dot{x} = f(x)$, x_0

\rightarrow Chaos : no closed-form solution

Forward Euler	$X_{k+1} = X_k + \Delta t f(X_k)$ $= [I + \Delta t A] X_k$
Backward Euler	$X_{k+1} = X_k + \Delta t f(X_{k+1})$ $= [I - \Delta t A]^{-1} X_k$



Accuracy & Stability

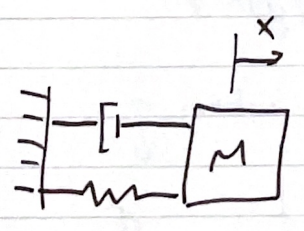
(J)

(S)

recall

Forward Euler : $X_{k+1} = M X_k$

$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots \rightarrow X_N$
 $X_0 \rightarrow M X_0 \rightarrow M^2 X_0 \rightarrow M^3 X_0 \rightarrow \dots \rightarrow M^N X_0$
 \hookrightarrow what kind matrix gives stability



$\ddot{x} + 2\zeta \omega_0 \dot{x} + \omega_0^2 x = 0$

$\zeta = \frac{c}{2\sqrt{km}}$ damping ratio

$\omega_0 = \sqrt{\frac{k}{m}}$ natural frequency

$\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}}_A \begin{bmatrix} x \\ v \end{bmatrix}$

\rightarrow use Forward Euler

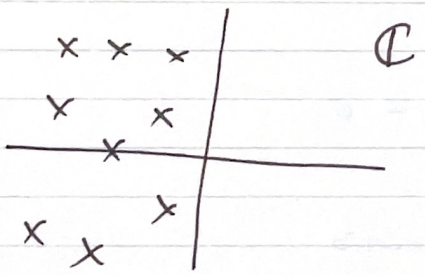
	FE	BE	ODE
Local Error	$O(\Delta t^2)$	$O(\Delta t^2)$	$O(\Delta t^5)$
Global Error	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t^4)$

Continuous ODEs

(Not exactly ODE)
discrete iteration

$$\dot{X} = AX$$

stable when:
 $\text{Re}(\text{eig}(A)) < 0$

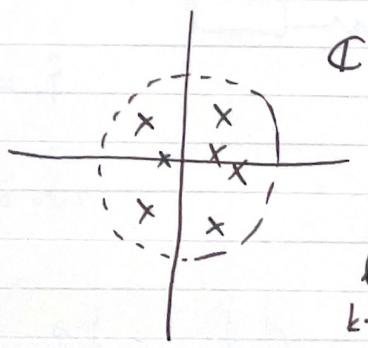


$$X_{k+1} = M X_k$$

stable when:
All $|\text{eig}(M)| < 1$
 $M = TDT^{-1}$

$$M^k = T D^k T^{-1}$$

stable when ↓



$$\lim_{k \rightarrow \infty} M^k = 0$$

$$M^k = T D^k T^{-1}$$

$$\begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \lambda_N^k \end{bmatrix}$$

Try $\lambda = 1.1 \rightarrow \infty$
Try $\lambda = 0.9 \rightarrow 0$

$$\lambda = R e^{i\theta}$$

$$\lambda^N = R^N e^{iN\theta}$$

Ex: $\dot{y} = \lambda y$ $y(0) = y_0$ Stable if $\text{Re}(\lambda) < 0$

Forward Euler: $y_{k+1} = (1 + \Delta t \lambda) y_k$: $y_0 \rightarrow y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_N$

Back. Euler: $y_{k+1} = (1 - \Delta t \lambda)^{-1} y_k$

$$y_N = (1 + \Delta t \lambda)^N y_0$$

$$y_N = (1 - \Delta t \lambda)^{-N} y_0$$

when is F.E stable iff

$$|1 + \Delta t \lambda| < 1$$

unstable if

$$|1 + \Delta t \lambda| > 1$$

assume $\text{Re}(\lambda) < 0 \rightarrow$ stable

despite being stable,

numerically, system could be unstable

when

$$|1 + \Delta t \lambda| > 1$$

e.g. $\Delta t \rightarrow$ too large.

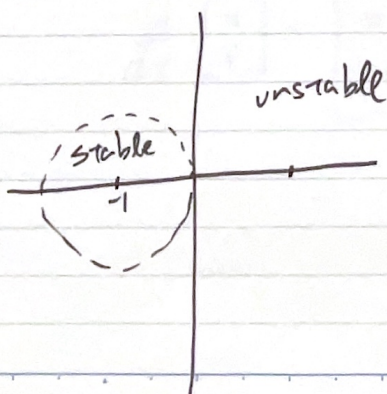
B.E stable iff

$$|(1 - \Delta t \lambda^{-1})| < 1$$

unstable if

$$|(1 - \Delta t \lambda^{-1})| > 1$$

F.E.



$\Delta t \lambda = z$

B.E.

Stable.

