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Quadrotor Trajectory Planning

Minimum Snap/Jerk w/ Bezier Curve

Why use Bezier Curve to parameterize trajectory? ↳ parameterized w/ Bernstein Basis, controlled by control points

1. endpoint interpolation property
   will not pass thru middle points
2. Convex Hull properties
   if all control points stay within the corridor → safety ensurance
3. Hodograph derivatives

↳ These could ensure the trajectory stays within constraints

why minimum snap, should refer to differential flatness, a concept that is used in non-linear control.

Differential Flatness (see later section for quadrotor flatness mapping)

for dynamical system

$$\dot{x} = f(x) + g(x)u \qquad f: R^n \to R^n \ \ x \in R^n, \quad g: R^n \to R^n \ \ u \in R^m, \quad rank(g) = m$$

we say that the system is differential flat if there exists

$\zeta \in R^m$ which can be determined by $x \in R^n$ $u \in R^m$

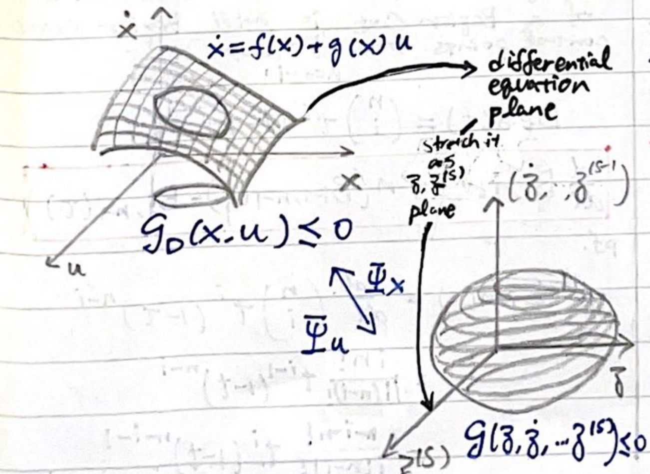
↓

differential outputs

$\zeta \in R^m$, could determine all state $x \in R^n$ input $u \in R^m$ through:

$$x = \Psi_x(\zeta, \dot{\zeta}, \cdots, \zeta^{(s-1)}),$$
$$u = \Psi_u(\zeta, \dot{\zeta}, \cdots, \zeta^{s})$$

Ψ_x, Ψ_u are transformation processes.



for quadrotor

$$\zeta = (r, \psi) \in R^3 \times SO(2)$$

$$\downarrow \Psi_x \ \Psi_u$$

$$x = \{r, v, R, \omega\} \in R^3 \times R^3 \times SO(3) \times R^3$$

$$u = \{f, T\} \in R_{\geq 0} \times R^3$$

detailed in later section

∴ so we can then plan a trajectory without differential constraints

Minimum Snap/Jerk

represent the trajectory as piecewise polynomial

$$f_j(t) = \sum_{i=0}^{n} p_i t^i \qquad j = 0 \ldots m$$

- by harnessing flatness, we optimize the trajectory
- optimize it w/ snap $^{(4)}$ || jerk $^{(3)}$

$$f^{(4)}(t) = \sum_{i \geq 4} i(i-1)(i-2)(i-3)\, t^{i-4} p_i$$

$$J(t) = \int_{T_{j-1}}^{T_j} \left(f^{(4)}(t)\right)^2 dt$$

$$= p_i p_l \sum_{\substack{i \geq 4 \\ l \geq 4}} \left[\begin{matrix} \vdots \\ \cdots \dfrac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7} t \\ \vdots \end{matrix} \right]_{T_{j-1}^{(i+l-7)}}^{T_j^{(i+l-7)}}$$

$$= p_i^T Q p_l$$

where
$$Q = \left[\begin{matrix} \vdots \\ \cdots \dfrac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7} T^{i+l-7} \cdots \\ \vdots \end{matrix} \right]$$

∴ $\min J(t)$ s.t. $Ap = b$ (continuous constraints)
$Ap \leq b$ (limitation constraints)

Minimum Snap/Jerk w/ Constraints.

Bernstein polynomials

from monomial basis → Bernstein basis

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$P_j(t) = c_j^0 b_n^0(t) + c_j^1 b_n^1(t) + \cdots + c_j^n b_n^n(t)$$

$$= \sum_{i=0}^{n} \underbrace{c_j^i}_{\text{control points}} \underbrace{b_n^i(t)}_{\text{bernstein basis}}$$

for n+1 points, polynomial has n order

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Berstein Polynomials (cont'd)

from above, any polynomials:

monomial basis $\rightarrow$ Bernstein basis

$$P = MC$$

recall min. snap $\quad \min J = \min P^T Q P$

here $\quad = \min C^T M^T Q M C$

$$P_j(t) = C_j^0 b_n^0(t) + C_j^1 b_n^1(t) + \cdots + C_j^n b_n^n(t)$$

$$= \sum_{i=0}^{n} C_j^i b_n^i(t) \qquad b_n^i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

let $f_u(t)$ be the whole trajectory

$$f_u(t) = \begin{cases} s_1 \sum_{i=0}^{n} C_{u_1}^i b_n^i\left(\frac{t-T_0}{s_1}\right), & t \in [T_0, T_1] \\ s_2 \sum_{i=0}^{n} C_{u_2}^i b_n^i\left(\frac{t-T_1}{s_2}\right), & t \in [T_1, T_2] \\ \vdots \\ s_m \sum_{i=0}^{n} C_{u_m}^i b_n^i\left(\frac{t-T_{m-1}}{s_m}\right), & t \in [T_{m-1}, T_m] \end{cases}$$

m^{th} peice trajectory

$$\tau = \frac{t - T_j}{s_{j+1}} \quad \text{normalization}$$

$$\tau = [0, 1]$$

$$J = \sum_{u \in \{x,y,z\}} \int_0^T \left(\frac{d^k f_u(t)}{dt^k}\right)^2 dt$$

on u axis, j^{th} trajectory $\qquad s\,d\tau = dt$

$$J_{u_j} = \int_0^{s_j} \left(\frac{d^k f_{u_j}(t)}{dt^k}\right)^2 dt$$

$$= \int_0^1 \left(\frac{s_j\, d^k g_{u_j}(\tau)}{s_j^k\, d\tau}\right)^2 s_j\, d\tau$$

$$= \int_0^1 \frac{s_j^2}{s_j^{2k}} \cdot s_j \left(\frac{d^k g_{u_j}(\tau)}{d\tau}\right) d\tau$$

$$= \int_0^1 s_j^{3-2k} \frac{d^k g_{u_j}(\tau)}{d\tau} d\tau$$

e.g. for $t \in [0, T]$

$$J = \int_0^T \left(\frac{d^k (s \cdot \sum C_i b^i(\frac{t}{s}))}{dt^k}\right)^2 dt$$

$$= \int_0^1 \left(\frac{d^k (s \cdot \sum C_i b^i(\tau))}{s^k d\tau^k}\right)^2 s\, d\tau$$

$$= \int_0^1 s^{3-2k} \left(\frac{d^k \sum C_i b^i(\tau)}{d\tau^k}\right)^2 d\tau$$

$$= \int_0^1 s^{3-2k} \left(\frac{d^k \sum P_i \tau^i}{d\tau^k}\right)^2 d\tau$$

get Q from $P^T Q P$ or $C^T M^T Q M C$

$$Q = \begin{bmatrix} & \vdots & \\ \cdots & \frac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7} s^{-2k+3} & \cdots \\ & \vdots & \end{bmatrix}$$

get M from $C^T M^T Q M C$

pascal triangle

now we have Q & M

minimize objectives: $\quad \min J = C^T M^T Q M C$

minimize conditions: $\quad Ac = b$

$\quad Ac \leq b$

$$c \in R^{(n\_order+1) \times m}$$

Bezier Curve Hodograph

Hodograph implies that the derivatives of a Bezier Curve is still Bezier curve

control points from P_{i} ($i=0 \sim n+1$) $\rightarrow Q_i$ ($i = 0 \sim n$)

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\boxed{\frac{d}{dt} B_{i,n}(t) = n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)}$$

pf.

$$\frac{d}{dt} B_{i,n}(t) = \frac{d}{d\tau} \binom{n}{i} \tau^i (1-\tau)^{n-i}$$

$$= \frac{i\, n!}{i!(n-i)!} t^{i-1} (1-t)^{n-i}$$

$$- \frac{(n-i)n!}{i!(n-i)!} t^i (1-t)^{n-i-1}$$

$$\frac{d}{dt}B_{i,n}(t)=\frac{i\,n!}{i!(n-i)!}t^{i-1}(1-t)^{n-i}-\frac{(n-i)\,n!}{i!(n-i)!}t^{i}(1-t)^{n-i-1}$$
$$=\frac{n(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}-\frac{n(n-1)!}{i!(n-k-1)!}t^{i}(1-t)^{n-i-1}$$
$$=n\left[\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}-\frac{(n-1)!}{i!(n-k-1)!}t^{i}(1-t)^{n-i-1}\right]$$
$$=n\left(B_{i-1,n-1}(t)-B_{i,n-1}(t)\right) ✠$$

$$F(t)=\sum_{i=0}^{n}B_{i,n}(t)P_i$$

$$\boxed{\frac{d}{dt}F(t)=\sum_{i=0}^{n-1}B_{i,n-1}(t)Q_i \quad Q_i=n(P_{i+1}-P_i)}$$

pf.

let $n=2$

$$F(t)=\sum_{i=0}^{2}B_{i,n}(t)P_i$$
$$=B_{0,2}(t)P_0+B_{1,2}(t)P_1+B_{2,2}(t)P_2$$

$$\frac{d}{dt}F(t)=2(\underbrace{\cancel{B_{-1,1}(t)}}_{=0}-B_{0,1}(t))P_0+2(B_{0,1}(t)-B_{1,1}(t))P_1+2(B_{1,1}(t)-\underbrace{\cancel{B_{2,1}(t)}}_{=0})P_2 \quad \text{—①}$$

when $n=1$, derivative of $F(t)$, $n=2$

$$F(t)=\sum_{i=0}^{1}B_{i,n}(t)Q_i$$
$$=B_{0,1}(t)Q_0+B_{1,1}(t)Q_1 \quad \text{—②}$$

from ①

$$\frac{d}{dt}F(t)=(-2P_0+2P_1)B_{0,1}(t)+(-2P_1+2P_2)B_{1,1}(t)$$
$$=2(P_1-P_0)B_{0,1}(t)+2(P_2-P_1)B_{1,1}(t)$$

from ②

$$=Q_0\,B_{0,1}(t)+Q_1\,B_{1,1}(t)$$

$$Q_0=2(P_1-P_0)$$
$$Q_1=2(P_2-P_1)$$

$$Q_i=n(P_{i+1}-P_i) ✠$$

e.g.

$P_0\ P_1\ P_2\ P_3\ P_4$ $n=4$ pts x5

$\frac{d}{dt}$ ↳ $n(P_1-P_0)$, $n(P_2-P_1)$, $n(P_3-P_2)$, $n(P_4-P_3)$ pts x4

$\frac{d}{dt}$ ↳ $(n-1)\begin{pmatrix} n(P_2-P_1)-n(P_1-P_0) \\ n(P_3-P_2)-n(P_2-P_1) \\ n(P_4-P_3)-n(P_3-P_2)\end{pmatrix}$ pts x3

Equality Condition

starting condition
ending condition
continuous condition

$$Ac=b$$

$c\in R_{(n\_order+1)\times m}$

n_order: highest order of polynomial

m: no. of trajectories

e.g. $n\_order=7$, $m=5$

$c\in R_{(7+1)\times 5}=R_{40}$

$A\in R_{d\times(n\_order+1)\times m}=R_{d\times 40}$

$b\in R_d \rightarrow \times d$ constraints

starting condition, ending condition $j=0$, $j=f$

$a^0_{i,j}=c_0\ (i,j)$ $\quad a^1_{i,j}=c_1$ $\quad a^2_{i,j}=c_2$

$P=a^0_{i,j}(s_j)^1$

$V=\frac{d}{dt}a^0_{i,j}(s_j)^0=n(a^1_{i,j}-a^0_{i,j})$

$a=\frac{d}{dt^2}a^0_{i,j}(s_j)^{-1}=n(n-1)[(a^2_{i,j}-a^1_{i,j})-(a^1_{i,j}-a^0_{i,j})]$ from above

↳ write everything into matrix

e.g. $n\_order=7$, $m=5$, $s_j=t$

$$Ac=b$$

$$\begin{bmatrix} 1t & 0 & 0 & \\ -7t^0, & 7t^0 & 0 & 0 \\ 42t^{-1}, & -84t^{-1}, & 42t^{-1} & \\ & 0 & & \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_8\end{bmatrix}=\begin{bmatrix} P \\ V \\ a \\ \vdots \\ 0\end{bmatrix}$$

continuous condition

$P_{n\_order+1,j}=P_{0,j+1}$ $\quad j=0,1,2,\ldots m$

$V_{n\_order+1,j}=V_{0,j+1}$ $\quad j=0,1,2,\ldots m$

$a_{n\_order+1,j}=a_{0,j+1}$ $\quad j=0,1,2,\ldots m$

$$\therefore \begin{cases} P_{n\_order+1,j}-P_{0,j+1}=0 \\ V_{n\_order+1,j}-V_{0,j+1}=0 \\ a_{n\_order+1,j}-a_{0,j+1}=0 \\ j \\ \vdots \\ s_{\cdot\cdot} \end{cases} =0 \longrightarrow$$

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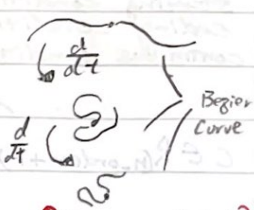
Equality condition (cont'd)

$$\begin{cases} P_{n\_order+1,j} - P_{0,j+1} = 0 \\ V_{n\_order+1,j} - V_{0,j+1} = 0 \\ a_{n\_order+1,j} - a_{0,j+1} = 0 \end{cases}$$



$$\Rightarrow \begin{cases} a^{n\_order+1}_{\mu_j} - a^{0}_{\mu_{j+1}} = 0 \\ \frac{d}{dt} a^{n\_order+1}_{\mu_j} - \frac{d}{dt} a^{0}_{\mu_{j+1}} = 0 \\ \frac{d^2}{dt^2} a^{n\_order+1}_{\mu_j} - \frac{d^2}{dt^2} a^{0}_{\mu_{j+1}} = 0 \end{cases}$$

$\left\{ \begin{array}{c} \text{first pt of } j+1 \\ \| \\ \text{last pt of } j \end{array} \right\}$

$$\Rightarrow \begin{cases} a^{n\_order+1}_{\mu_j} - a^{0}_{\mu_{j+1}} = 0 \\ n(a^{n\_order+1}_{\mu_j} - a^{n\_order}_{\mu_j}) - n(a^{1}_{\mu_{j+1}}, a^{0}_{\mu_{j+1}}) = 0 \\ n(n-1)\left[(a^{n\_order+1}_{\mu_j} - a^{n\_order}_{\mu_j}) - (a^{n\_order}_{\mu_j} - a^{n\_order-1}_{\mu_j})\right] \\ -n(n-1)\left[(a^{2}_{\mu_{j+1}} - a^{1}_{\mu_{j+1}}) - (a^{1}_{\mu_{j+1}} - a^{0}_{\mu_{j+1}})\right] = 0 \end{cases}$$

↳ write everything into matrix

e.g.
$n\_order = 7$
$m = 5$ — 4 connection conditions
condition @ $j = 1, 2$

↓ 12 conditions

$$\begin{bmatrix} \cdots & & 1 & -1 & & \cdots \\ \cdots & -7 & 7 & 7 & -7 & \cdots \\ 42 & -84 & 42 & -42 & 84 & -42 \\ \cdots & & & & & \cdots \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_8 \\ \vdots \\ c_{40} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$R_{12\times40}$ R_{40} R_{12}

w, w/o t^{-n} does not matter as $Ac=0$

$$\therefore Ac = b$$

$A \in R_{d\times40}$
$c \in R_{40}$
$b \in R_d$

d: no. of conditions
starting
ending
continuous

Inequality condition (cont'd)

safety corridor
dynamic constraints

$$Ac \le b$$

- main reason for us to use bezier curve is that we can CONFINE all control points within our desired range
- all expression should be in the form of $Ac \le b$

$A \in R_{d\times[(n\_order+1)\times m]}$ — no. of trajectories; no. of ctrl pts per traj

$c \in R_{(n\_order+1)\times m}$

$b \in R_d$ $d \Rightarrow$ no. of conditions

safety corridor

let $a \le x \le b$
$c \le y \le d$
$e \le z \le f$

$$\Rightarrow \begin{cases} x \le b \\ -x \le -a \\ y \le d \\ -y \le -c \\ z \le f \\ -z \le -e \end{cases}$$

write everything in matrix

$$Ac \le b$$

express everything in terms of control points

$$\Rightarrow t^{1} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ -1 & & & \\ & -1 & & \\ & & -1 & \ddots \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \le b$$

ieq_P

$$\Rightarrow t^{0} \begin{bmatrix} -7 & 7 & & \\ & -7 & 7 & \\ & & & \ddots \\ 7 & -7 & & \\ & 7 & -7 & \ddots \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \le b$$

ieq_V

$$\Rightarrow t^{-1} \begin{bmatrix} 42 & -84 & 42 & & \\ & 42 & -84 & 42 & \\ & & & & \ddots \\ -42 & 84 & -42 & & \\ & -42 & 84 & -42 & \ddots \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \le b$$

ieq_V

e.g. $n\_order = 7$
$m = 5$

no. of P ctrl_pts $= 80$
V $= 70$
a $= 60$

$\therefore A \in R_{210\times40}$
$\therefore c \in R_{40}$
$b \in R_{210}$