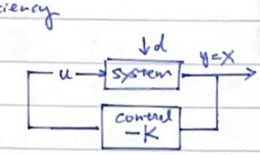
Control Bootcamp

- passive control
- active control
  - open loop
  - close loop (w/ feedback)
    1. tackle w/ uncertainty
    2. tackle w/ instability
    3. tackle w/ disturbances
    4. efficiency

- $\dot{x} = Ax + Bu$

  $x(t) = e^{At}x(0)$
- $y = Cx$



$\Rightarrow u = -Kx$

$\dot{x} = Ax - BKx$

$= \underline{(A - BK)}x$

determine K to make it stable

Linear System

$\dot{x} = Ax \quad x \in R^n \quad x(t) = e^{At}x(0)$

$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$

from below

$= e^{TDT^{-1}t}$

$= TT^{-1} + TDT^{-1}t + \frac{TDT^{-1}TDT^{-1}}{2!}t^2 + \cdots$

$= T[I + Dt + \frac{D^2t^2}{2!} + \cdots +]T^{-1}$

$= \boxed{Te^{Dt}T^{-1}}$

Eigenvalues & Eigenvectors : $AT = TD$ >> [T, D] = eig(A);

$A\xi = \lambda\xi$

$T = [\xi_1, \xi_2, \cdots \xi_n]$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$\Rightarrow T^{-1}AT = D$, let $\boxed{x = Tz}$, $\dot{x} = T\dot{z} = Ax$

$\Rightarrow \dot{x} = T\dot{z} = Ax$

$\Rightarrow T\dot{z} = ATz$

$\Rightarrow \dot{z} = T^{-1}ATz$

$\Rightarrow \boxed{\dot{z} = Dz}$

$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

in sum $z(t) = e^{Dt}z(0)$

$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_n t} \end{bmatrix} z(0)$

$\therefore \underbrace{\underbrace{Te^{Dt}\underbrace{T^{-1}x(0)}_{z(0)}}_{z(t)}}_{x(t)}$, i.e. $x(t) = Te^{Dt}T^{-1}x(0)$

~~Stability~~
Eigenvalues

Δ recall $\dot{x} = Ax \quad x \in R^n$

$[T, D] = eig(A);$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \ e^{\lambda_n t} \end{bmatrix}$

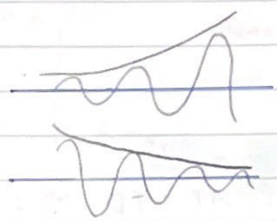
$\left.\right\} \quad x(t) = T e^{Dt} T^{-1} x(0)$

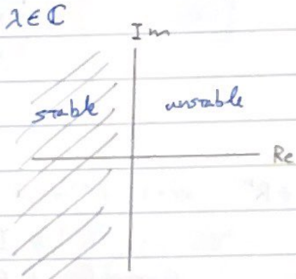
Δ $\lambda = a + bi$

$e^{\lambda t} = e^{at}\left[\cos(bt) + i\sin(bt)\right] \qquad \lambda \in \mathbb{C}$

if $a > 0$

if $a < 0$





(continuous case)

Δ discrete system

$x_{k+1} = \tilde{A} x_k \quad , \quad x_k = x(k\Delta t)$

$\underset{\text{discrete}}{\tilde{A}} = e^{\overset{}{A}_{\text{continuous}}\Delta t}$

$\tilde{A}\tilde{T} = \tilde{T}\tilde{D}$

$\tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$

$x_1 = \tilde{A}x_0 \qquad = \tilde{T}\tilde{D}\tilde{T}^{-1}x_0$

$x_2 = \tilde{A}x_1 = \tilde{A}^2 x_0 \qquad = \tilde{T}\tilde{D}^2\tilde{T}^{-1}x_0$

$x_3 = \tilde{A}^3 x_0 \qquad = \tilde{T}\tilde{D}^3\tilde{T}^{-1}x_0$

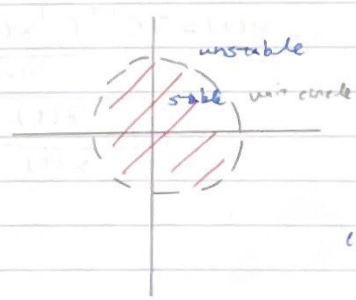
$\vdots \qquad\qquad \vdots$

$x_n = \tilde{A}^n x_0 \qquad = \tilde{T}\tilde{D}^n\tilde{T}^{-1}x_0$

recall $\tilde{D}$ is a diagonal matrix

w/ $\lambda = a + bi = Re^{i\theta}$

$\lambda^n = R^N e^{iN\theta}$



(discrete case)

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Linearizing Around a Fixed Point

△ from non-linear $\dot{x} = f(x) \Rightarrow$ to linear $\dot{x} = Ax \quad x \in \mathbb{R}^n$

1. find fixed pts
   $\bar{x}$ s.t. $f(\bar{x}) = 0$

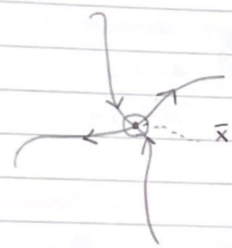
2. linearize about $\bar{x}$

$$\left.\frac{Df}{Dx}\right|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]_{ij}$$

e.g. $\dot{x}_1 = f_1(x_1, x_2) = x_1 x_2$

$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{Df}{Dx} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$

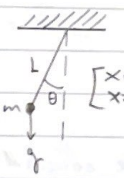


$\bar{x} = 0$ (shift w/ $\bar{x}$) assumed

$$\dot{x} = f(x)$$
$$= f(\bar{x}) + \left.\frac{Df}{Dx}\right|_{\bar{x}} \cdot (x - \bar{x}) + \underbrace{\frac{D^2 f}{Dx^2} \cdot (x - \bar{x})^2 + \cdots}_{\text{too small, neglected}}$$

$$\Delta \dot{x} = \left.\frac{Df}{Dx}\right|_{\bar{x}} \Delta x \Longrightarrow \Delta \dot{x} = A \Delta x$$
$$\dot{x} = Ax$$

e.g. Pendulum

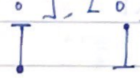


$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \delta\dot{\theta}$, $\frac{g}{L} = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix}$$

1. F.P. $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (case 1), $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$ (case 2)



$$\frac{Df}{Dx} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\delta \end{bmatrix}$$

case 1 $A = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$ $\lambda = \begin{matrix} -0.05 + 0.9987i \\ -0.05 + -0.9987i \end{matrix}$ (stable locally)

case 2 $A = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}$ $\lambda = \begin{matrix} -1.0512 \\ 0.9512 \end{matrix}$ $\begin{pmatrix} \text{stable} \\ \text{unstable} \end{pmatrix}$ saddle point.

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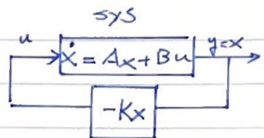
Controllability

affect stability

△ recall: $\dot{x} = \underset{}{A}x \quad x \in R^n$

△ now, w/ control input:

$$\dot{x} = \underset{R^{n\times n}}{A}x + \underset{R^{n\times k}}{B}\,\underset{R^k}{u}$$



"optimal" for linear system.

when $u = -Kx$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{by choosing } K,\ \text{we can change the "dynamics"}}x$$

△ intuition: if my system is controllable: I can put the eigenvalues of A-BK) "anywhere I want", also indicating that x can be "anywhere in R^n"

→ when or how to know?

△ >> ctrb(A, B)

• $\dot{x} = Ax$

* $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ ← uctrb: as x_1 is not coupled w/ x_2 or u, making it impossible to twist anything on it

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ ← ctrb

• rather not obvious:

** $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ ← ctrb: x_1 & x_2 are coupled. twisting just one u, can do things on x_1, x_2 simultaneously.

• $\mathcal{C} = [B \quad AB \quad A^2B \ \dots \ A^{n-1}B]$ i.f.f rank($\mathcal{C}$) = n then sys is ctrb

*: $\mathcal{C}_* = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ → uctrb (rank = 1)

**: $\mathcal{C}_{**} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ → ctrb (rank = 2)

• doing SVD on $\mathcal{C}$ gives you the controllability extent on each state
(most → least)
(singular vectors)

△ Note that the controllability discussed here is "linear controllable".

Controllability / Reachability / Eigenvalue Placement

△ recall: $\dot{x} = Ax + Bu$

$\mathcal{C} = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$

$\text{rank}(\mathcal{C}) = n \underset{\text{i.f.f.}}{\iff}$ ctrb

```
>> rank(ctrb(A,B));
```

△ Equivalences:

1. system is ctrb
2. Arbitrary eigenvalue (pole) placement

   $u = -kx \Rightarrow \dot{x} = \underbrace{(A - BK)}_{\text{arbitrary eigenvalue!}} x$ `>> K=place(A,B,eigs);`

3. Reachability (full) in R^n (can reach any vector in R^n given some u)

   Reachable set $R_t = \{ \xi \in R^n \mid \text{there is an input } u(t) \text{ s.t. } x(t) = \xi \}$

   $R_t = \mathbb{R}^n$

Controllability & Discrete-Time Impulse Response

△ recall: $\dot{x} = Ax + Bu \quad x \in R^n$

$C = [B \ AB \ A^2B \cdots A^{n-1}B]$

△ $x_{k+1} = \tilde{A}x_k + \tilde{B}u_k$ — Impulse Response

assume $x_0 = 0$ $u_0 = 1$

then $x_1 = \tilde{B} = \tilde{A}\cdot 0 + \tilde{B}\cdot 1$ $u_1 = 0$

$x_2 = \tilde{A}\tilde{B} = \tilde{A}\cdot\tilde{B} + \tilde{B}\cdot 0$ $u_2 = 0$

$x_3 = \tilde{A}^2\tilde{B} = \tilde{A}\cdot\tilde{A}\tilde{B} + \tilde{B}\cdot 0$ $u_3 = 0$

$\vdots$

$x_m = \underbrace{A^{m-1}B}$ $u_m = 0$

if this can "hit" or "affect" all axis in R^n } just an intuition

No

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not yes or no. not binary.
"to what extent"

Degrees of Controllability & Gramians

△ How controllable are different directions on R^n

△ $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

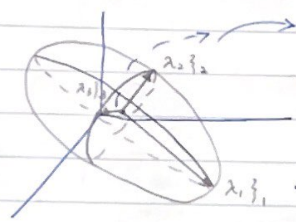
△ controllability Gramian

$$W_t = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau \quad \in R^{n \cdot n}$$

$W_t \xi = \lambda \xi$ larger eigenvalues, more controllable

in discrete time

$W_t \approx \mathcal{C}\mathcal{C}^T \longleftrightarrow$ svd of $\mathcal{C}$: $[U, \Sigma, V] = \text{svd}(\mathcal{C}, \text{'econ'})$



singular vectors of my controllability axis

△ stabilizability

stab. iff all unstable eigenvectors of A are in ctrb. space
lightly damped

(as if x is large, it is impossible to let each direction be controllable.

PBH Test (Popov - Belevitch - Hautus)

△ (A, B) is ctrb i.f.f

$\text{rank} [\underset{R^{n \cdot n}}{(A - \lambda I)}\ B] = n \quad \forall \lambda \in \mathbb{C}$

1. $\text{rank}(A - \lambda I) = n$ except for eigenvalues (of A)

   ∴ just need to perform PBH test @ λ

2. B needs to have some component in each eigenvector direction

3. if B is a random vector, i.e., B = randn (n, 1)

   ∴ (A, B) will be ctrb w/ high probability

   (as it is hard to generate vector w.o. all components on eigenvector direction

4. $\text{rank} [(A - \lambda I)\ B]$

   △ 2x multiple λ 2 columns → composite ctrb rank
   3x 3
   4x " 4
   ⋮ ⋮

△ or degenerate eigenvalues
(two values are close

Caley - Hamilton Theorem

△ Every matrix A satisfies its own characteristics (eigen value) equation

$$\det(A-\lambda I) = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

$$\Rightarrow A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + A\lambda + a_0 I = 0$$

(almost true for all A)

$$\Rightarrow A^n = -a_0 I - a_1 A - a_2 A^2 - \cdots - a_{n-1}A^{n-1}$$

$$\Rightarrow A^{2n} = \sum_{j=0}^{n-1} \alpha_j A^j \quad \text{(could be expressed as } n-1 \text{ or lower terms)}$$

△ $\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n$

$$e^{A\tau} = I + A + \frac{A^2\tau^2}{2}\cdots$$

$$\Rightarrow \quad = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \cdots + \alpha_{n-1}(t)A^{n-1}$$

not infinite

Reachability and controllability w/ Cayley-Hamilton

△ Reachability

if $\xi \in \mathbb{R}^n$ is reachable

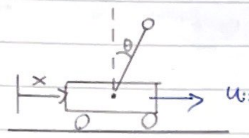
then $\xi = \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$ (for some $u(\tau)$)

$$\xi = \int_0^t \left(\phi_0(t-\tau)u(\tau) I B + \phi_1(t-\tau)u(\tau) AB + \cdots + \phi_{n-1}(t-\tau)u(\tau)A^{n-1}B \right) d\tau$$

$$= B\int_0^t \phi_0(t-\tau)u(\tau)\,d\tau + AB\int_0^t \phi_1(t-\tau)u(\tau)\,d\tau + \cdots + A^{n-1}B\int_0^t \phi_{n-1}(t-\tau)u(\tau)\,d\tau$$

$$= \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \left.\begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)\,d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)\,d\tau \\ \vdots \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)\,d\tau \end{bmatrix}\right\} n \times q$$

Inverted Pendulum



u = force on the cart in x direction

$$X = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \frac{d}{dt}X = f(X) \xRightarrow{\left.\frac{Df}{DX}\right|_{X_{fix}}} \dot{X} = AX + Bu$$

fixed points : $\theta = 0, \pi$

$\dot{\theta} = 0$

$\dot{x} = 0$

x free $\qquad \dot{X} = (A - BK)X$

refer to matlab

Pole placement

- >> K = place (A, B, eigs)
  >> eig (A − BK) = eigs
- try to design K such that [A−BK] matrix has stable poles
  # refer to matlab.

LQR

- >> K = place (A, B, eigs)
- where are the best eigs ?

Linear Quadratic Regulator (LQR) ∼ $O(n^3)$

- $J = \int_0^\infty (x^T Q x + u^T R u)\, dt$

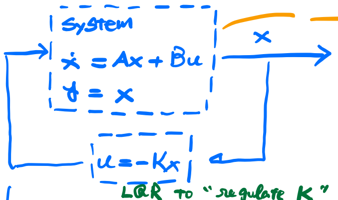
$$Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix} \qquad R = \begin{bmatrix} 0.001 \end{bmatrix}$$

- >> K = lqr (A, B, Q, R)

Motivation for Full-state estimation

Recall

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^q$



LQR to "regulate K"

$\dot{x} = Ax - BKx$

$\dot{x} = (A - BK)\,x$

(placing eig-values)

I don't necessarily hv all states in real life

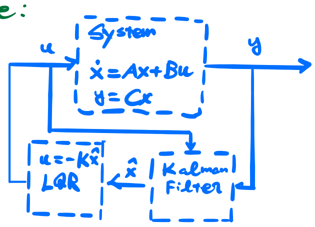
$\dot{x} = Ax + Bu$ (controllability) ctrb (A, B)

$y = Cx$ (observability) obsv (A, C)

Main Question here:

Can I estimate any state $\underline{x}$ from measurement $y(t)$

hence:



Observability

- Duality exists between $\begin{matrix} AB \\ AC \end{matrix}$
- observability matrix

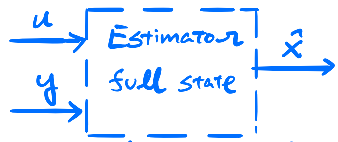
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

1. observable if
   `>> rank ( obsv (A, C)) = n`
2. can estimate x from y
3. `>> [U, Σ, V] = svd (O)`
   observability Gramian

V^T

In some direction, we hv higher signal to noise

Full State Estimation



dyn. system (linear)

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + \underbrace{K_f\,(y - \hat{y})}_{\text{update}}$$

filter

$$\hat{y} = C\hat{x}$$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K_f y - K_f C\hat{x}$$

$$= (A - K_f C)\hat{x} + \begin{bmatrix} B & K_f \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix}$$

↓

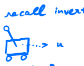
pick K_f to place the eigen values to hv optimal choice

Kalman filter

- w_d - Gaussian
  V_d Variance
  w_n - Gaussian
  V_n Variance
  &
- recall
  $\dot{\varepsilon} = (A - K_f C)\,\varepsilon$
  $\varepsilon = x - \hat{x}$
- cost function
  $J = \mathbb{E}\left((x-\hat{x})^T(x-\hat{x})\right)$
- `>>> Kf = lqe (A, C, Vd, Vn)`

Observability Example

- recall inverted pendulum



$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$

$\dot{x} = Ax + Bu$

$y = Cx$

`>> obsv (A, C)`

$C = [1\ 0\ 0\ 0]$

- subsystem : without x[0]

$x = \begin{bmatrix} \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$

with different measurement

$C = [1\ 0\ 0]$ meas. $\dot{x}$

$C = [0\ 1\ 0]$ meas. θ

$C = [0\ 0\ 1]$ meas. $\dot{\theta}$

→ `det ( gram (sys, 'o'))`
(`sys = ss (A, B, C, D)`)

y	det Wo
$\dot{x}$	50
θ	.03
$\dot{\theta}$	.03

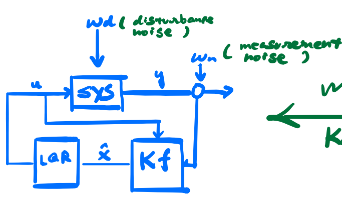
easier to observe w/ $\dot{x}$

Kalman filter ←

real system:

$\dot{x} = Ax + Bu + w_d$

$y = Cx + w_n$



← motivation of Kalman filter

↓

get the best K_f to place poles (eigs) based on w_d & w_n

Error $\varepsilon = x - \hat{x}$

$$\frac{d}{dt}\varepsilon = \frac{d}{dt}x - \frac{d}{dt}\hat{x}$$

$$= Ax + \cancel{Bu} - A\hat{x} + K_f C\hat{x} - K_f y - \cancel{Bu}$$

$$= Ax - A\hat{x} + K_f C\hat{x} - K_f \underline{y}^{\,Cx}$$

$$= A(x - \hat{x}) + K_f C(\hat{x} - x)$$

$$= A(x - \hat{x}) - K_f C(x - \hat{x})$$

$$= \underbrace{(A - K_f C)}\,\varepsilon$$

if observable, then place eigs by choosing K_f: so that error converge eventually