△ Inverse Kinematics

- Recall: forward kinematics

$T_{EE}(q) = \begin{bmatrix} C_{IE}(q) & {}_I r_{IE}(q) \\ 0 & 1 \end{bmatrix} \in SE(3)$

parametrize

$\chi_e = \begin{pmatrix} \chi_{e,P}(q) \\ \phi_e(q) \end{pmatrix} = f(q)$

$\chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix}$

- Q: $q = f^{-1}(\chi_E)$? give you χ_E, give me q

△ sol^n

- Analytic:
  - for 3 intersecting neighbouring axes
- Geometric:
  - use length, ... "geometric info"
- Algebraic
  - use TFs to get joints
- numerical!

△ numerical method: inverse differential kinematics

- recall

$w_e = J_{e0}\dot{q}$

$\dot{q} = J_{e0}^{+} w_e^*$ — pseudo inverse: multiple sol, singular

- singularities
  - occur @ q_s ⇒ $J_{e0}(q_s)$ is column deficient
  - $\dot{q} = J_{e0}^{+} w_e^*$ — inf ← column deficient ← small
- categories
  - boundary: when robot arm is @ its limitation, easy to prevent
  - internal: hard to prevent
- damped version of Moore-Penrose pseudo inverse

$\dot{q} = J^{+}w \Leftrightarrow \arg\min_{\dot{q}} \| J\dot{q} - w \|_2$

$\Leftrightarrow \arg\min_{\dot{q}} \| J\dot{q} - w\|_2^2 + \lambda \|w\|_2^2$

$\dot{q} = J^T(JJ^T + \lambda^2 I)^{-1} w$

- Redundancy

$w = J\dot{q}$

- $\dot{q} \in R^n$
- $w \in R^m$
- $J \in R^{m \times n}$ — $m > n$ → redundancy

- $J_{eo}\dot{q} = w_e^*$

⇒ $J_{eo}(J_{eo}^{+} w_e^* + N\dot{q}_0) = w_e^*$

⇒ $\dot{q} = J_{eo}^{+} w_e^* + N\dot{q}_0$

$N = N(J_{eo})$ null space

$J_{eo}N = 0$

- get N ?
  - $N = I - J_{eo}^{+} J_{eo}$
  - SVD
  - QR

  end-up w/ different basis that span null space

△ multi-task control

- track desired position & orientation, ensure kinematics
- break down tasks; $\{J_i, w_i^*\}$

$\dot{q} = \begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^{+} \begin{bmatrix} w_1^* \\ \vdots \\ w_{n_t}^* \end{bmatrix}$ — $\bar{J}$, $\bar{w}$

doesn't necessarily mean the end-effector's state space. It's just an objective space. J is the mapping

⇒ $\min \| \bar{J}\dot{q} - \bar{w} \|_2$

⇒ $\min \|\dot{q}\|_2^2$ s.t. $\bar{J}\dot{q} = \bar{w}$

- weighting $[w_1, \dots, w_n]$

$\bar{J}^{+W} = (\bar{J}^T W \bar{J})^{-1} \bar{J}^T W$

- Prioritization

recall: $\dot{q} = J_1^{+} w_1^* + N_1 \dot{q}_0$

⇒ $w_2 = J_2\dot{q} = J_2(J_1^{+} w_1^* + N_1\dot{q}_0)$ — only do things in null space

⇒ $\dot{q}_0 = (J_2N_1)^{+}(w_2^* - J_2J_1^{+}w_1^*)$

⇒ $\dot{q} = J_1^{+}w_1^* + N_1(J_2N_1)^{+}(w_2^* - J_2J_1^{+}w_1^*)$ — $\dot{q}_0$ → $J_2\dot{q} \cong w_2$

∴ $\dot{q} = \sum_{i=1}^{n_t} N_i \dot{q}_i$

w/ $\dot{q}_i = (J_i N_i)^{+}(w_i^* - J_i \sum_{k=1}^{i-1} N_k \dot{q}_k)$

△ Back to inverse kinematics

- $\dot{\chi}_E = J_{eA}(q)\dot{q}$ — analytical: always depends on parameterization
- $w = J_0 \dot{q}$ — geometric
- now $\Delta \chi_e = J_{eA}(q)\Delta q$
- tracking a point χ_e^*, $q_0 = q^0$

pseudo-code

```
q ← q^0
while ||χ_e* − χ_e(q)|| ≥ tol do
    J_eA ← ∂χ_e/∂q (q)
    J_eA^+ ← (J_eA)^+
    Δχ_e ← χ_e* − χ_e(q)
    q ← q + J_eA^+ Δχ_e
```

(usually joint-space bigger than task-space → redundant system)

△ iterative method

- problem 1

$q^{i+1} = q^i + J_{eA}^{+}\Delta X$ — if Δx ↑ & J_{eA} is ill-conditioned

∴ $q^{i+1} = q^i + k J_{eA}^{+}\Delta X$ — slower convergence

- problem 2

Jacobian is rank deficient → bad condition

∴ $q^{i+1} = q^i + J_{eA}^T(J_{eA}J_{eA}^T + \lambda^2 I)^{-1}\Delta X$

or $q^{i+1} = q^i + \alpha J_{eA}^T \Delta X$

△ Orientation

- depends on parameterization
- $\in SO(3)$
- $\Delta \chi_{rotate} = \Delta \varphi$

$C_{GS}(\Delta\varphi) = C_{GI}(\varphi^*) C_{GI}^T(\varphi^t)$

⇒ $q \leftarrow q + k_{pR} J_{eoR}^{+} \Delta\varphi$ — use geometric one (rotation vector)

△ Traj. ctrl.

- given $r_e^*(t)$, $\omega_e^*(t)$
- feedback
  - $\Delta r_e^t = r_e^*(t) - r_e(q^t)$
  - $\dot{q}_P^* = J_{eoP}^{+}(q^t)\cdot(\dot{r}_e^*(t) + k_{pP}\Delta r_e^t)$
  - $\dot{q}_R^* = J_{eoR}^{+}(q^t)\cdot(\omega(t)_e^* + k_{pR}\Delta\phi)$

△ L.A. revisit

△ determinants

- $\det(A) = 0$ — squeeze original state space, to lower dimension
- $\det(A) = c$ — if $\det(A) = c$, $Area = c$
  - if: $sign(c) = (-)$, then: "flipped"

$Ax = b$

- if $\det(A) = 0$:
  - A transform x to "lower dimension", cannot unpack. i.e. ∄ transform back
  - sometimes can! already "live" on the new "lower dim"
- if $\det(A) \neq 0$:
  - then can reverse left!

△ Rank

$rank(A) = n$

then n is the dimension of the spanned space of the output

column space = all possible outputs Ax

$Ax = b$ — DoF, of the "column space" dimension

△ Nullspace (kernel)

$Ax = 0 \quad A \in R^{m \times n}$

↳ a set of vectors, which form a space, whose dots get "squeeze" to the origin;

note that all A has trivial null space $\{0\}$;

null space $\in R^{n-m}$ if $n > m$; reduced rank

△ ill/well-condition of matrix

$Ax = c$

ill conditioned? / well conditioned? } small change in $[A]$, small/large change in $[c]$?

vice-versa → small change in $[c]$, small/large change in $[A]$?

- △ small ΔA → small Δc "well-conditioned"
- △ small ΔA → large Δc "ill-conditioned"

e.g. $\begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$, $x = \begin{bmatrix} -100 \\ -200 \end{bmatrix}$

$\begin{bmatrix} 401 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$, $x = \begin{bmatrix} 40000 \\ 79800 \end{bmatrix}$

△ condition number

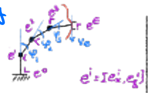
cond$[A] \approx 1.0$ well

cond$[A] > 1.0$ ill

cond$[A] = \|A\|_\infty \|A^{-1}\|_\infty$

Rule of thumb: $\log[\text{cond}[A]]$ = decimals of accuracy lost

△ inverse kinematics case study



(1) given $q_t = [\pi/6, \pi/3, \pi/3]^T$, ${}^0 r^*_{E,t} = [1, 1]^T$, $\dot{e} = [\dot{e}_x, \dot{e}_y]$

find $\dot{q}$ (generalized velocity)

$w = J\dot{q}$ (assume $l_i = 1$)

$r_E = \begin{bmatrix} S\theta_1 + S(\theta_1+\theta_2) + S(\theta_1+\theta_2+\theta_3) \\ 1 + C\theta_1 + C(\theta_1+\theta_2) + C(\theta_1+\theta_2+\theta_3) \end{bmatrix}$

⇒ $J_E = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \frac{\partial r_1}{\partial q_3} \\ \frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \frac{\partial r_2}{\partial q_3} \end{bmatrix}$

= [illegible]

⇒ w/ $q = [\pi/6, \pi/3, \pi/3]^T$

$J_E = \frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix}$

⇒ $\dot{q} = J_E^{+} w$

$= J^T(JJ^T)^{-1}[\,]$

$= \begin{bmatrix} 0.069 \\ -0.56 \\ -0.575 \end{bmatrix}$

check! $\frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 0.069 \\ -0.56 \\ -0.575 \end{bmatrix} = [\,]$

(2) multi-task

we want $\dot{q}_1 = \dot{q}_2 = 0$

$J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- parallel (no priorities)

$\dot{q} = \begin{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}^{+} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $R^{3\times 1}$

$[R^{4\times 3}]^{+}[R^{4\times 1}]$

⇒ $\dot{q} = \begin{bmatrix} -0.133 \\ -0.067 \\ -1.132 \end{bmatrix}$

check! $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\dot{q} = \begin{bmatrix} 1.039 \\ 1.483 \\ -0.133 \\ -0.067 \end{bmatrix} \approx \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

(3) w/ priority, task 1 most important! "$w = J\dot{q}$"

⇒ $\dot{q} = J_1^{+}w_1 + N_1\dot{q}_0$ — $I - J_1^{+}J_1$

⇒ $J_2\dot{q} = J_2J_1^{+}w_1 + J_2N_1\dot{q}_0 = w_2$

⇒ $\dot{q}_0 = (J_2N_1)^{+}(w_2 - J_2J_1^{+}w_1)$

⇒ $\dot{q} = J_1^{+}w_1 + N_1 \dot{q}_0$

$= \begin{bmatrix} -0.169 \\ -0.085 \\ -1.070 \end{bmatrix}$

$J_1[I - J_1^{+}J_1] = J_1 - J_1J_1^{+}J_1 = 0$

check! $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\dot{q} = \begin{bmatrix} \vdots \\ -0.169 \\ -0.085 \end{bmatrix}$

△ matrix norm

$[A] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\|A\|_\infty = \max\{|1|+|2|, |0|+|1|\}$ — row sum w/ abs.

$= \max\{3, 1\} = 3$

$\|A\|_1 = $ sum of row 1 $= 3$

$\|A\|_2 = $ sum of row 2 $= 1$

e.g. recall cond$[A] = \|A\|_\infty \|A^{-1}\|_\infty$

$A = \begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1.003 & -5.01 \\ \vdots & \vdots \end{bmatrix}$

⇒ $\|A\|_\infty = |-801| + |401| = 1201$

$\|A^{-1}\|_\infty = |-2| + |-1| = 3$

⇒ cond$[A] = 3603$

△ more on matrix norm

- recall vector norms
  - L_1-norm: $\|u\|_1 = |u_1| + |u_2|$
  - L_2-norm: $\|u\|_2 = \sqrt{u_1^2 + u_2^2}$
- matrix norms "based on" vector norm

matrix → linear transformation

∴ matrix norm: "HOW BIG MY MATRIX CAN GET MY OUTPUT VECTOR".

$\|A\|_2 = \max_{\|u\|_2 = 1} \|Au\|_2$

- $\|AB\|_p \leq \|A\|_p \|B\|_p$
- $\|A^k\|_p = \max_{\|u\|_p = 1} \|A^k u\|_p$

$A[\dots] = \lambda_i[\dots]$

$Au = \lambda u$

⇒ $\|Au\|_p = |\lambda_i| \|u\|_p \leq \|A\|_p \|u\|_p$

⇒ $|\lambda_i| \leq \|A\|_p$

⇒ $\max|\lambda_i| \leq \|A\|_p$

⇒ $\max|\lambda|^k \leq \|A^k\|_p$

△ Dynamics

this chapter basically talks about Both description of motion & source of motion

- kinematic v.s. dynamic
  - don't care about the cause, let me know the relation! / where's the source
- cause of motion

τ → $\ddot{q}$: cause → description of motion!

※ $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau + J_c^T F_c$

- Newton-Euler (free cut, conservation of impulse & angular momentum)
- Projected Newton-Euler (generalized coordinates)
- Lagrange II (energy)

△ Principle of virtual work (D'Alembert's Principle)

- Dynamic equilibrium ⇒ 0 virtual work

$\delta W = \int_B \delta r^T (\ddot{r}dm - dF_{ext}) = 0$

- Newton's law for every particle

$F = ma$

⇒ $F = \frac{d}{dt}(mv) = \dot{P}$ — P: momentum (linear impulse)

momentum

$r \times F = r \times ma$

$\Gamma = \frac{d}{dt}(r \times mv) = \dot{N}$ — N: angular momentum

△ Derivation of virtual displacements of Single Rigid body

- from rigid body kinematics

$r = r_{AS} + P$ — $[\Omega]_\times P = -[P]_\times \Omega$

$\dot{r} = v_S + \Omega \times P = [I_{3\times3} \; -[P]_\times]\begin{bmatrix} v_S \\ \Omega \end{bmatrix}$

$\ddot{r} = a_S + \dot{\Psi} \times P + \Omega \times (\Omega \times P)$

$= [I_{3\times3} \; -[P]_\times]\begin{bmatrix} a_S \\ \dot{\Psi} \end{bmatrix} + [\Omega]_\times[\Omega]_\times P$

$\delta r = \delta r_S + \delta\Phi \times P$

$= [I_{3\times3} \; -[P]_\times]\begin{bmatrix} \delta r_S \\ \delta\Phi \end{bmatrix}$

- Apply principle of virtual work

$\delta W = \int_B \delta r^T(\ddot{r}dm - dF_{ext}) = 0$

虚功 work 功 → force × distance

$\int_B \begin{pmatrix}\delta r_S \\ \delta\Phi\end{pmatrix}^T \begin{bmatrix} I_{3\times3} \\ [P]_\times \end{bmatrix}$ (δr)

$\cdot \left([I_{3\times3} \; -[P]_\times]\begin{bmatrix} a_S \\ \dot\Psi \end{bmatrix} + [\Omega]_\times[\Omega]_\times P \right) dm - dF_{ext}$

$= \begin{pmatrix}\delta r_S \\ \delta\Phi\end{pmatrix}^T \int_B \left(\begin{bmatrix} I_{3\times3}dm & -[P]_\times dm \\ [P]_\times dm & -[P]_\times[P]_\times dm \end{bmatrix}\begin{pmatrix} a_S \\ \dot\Psi \end{pmatrix} + \begin{bmatrix} [\Omega]_\times^2 P dm \\ [P]_\times[\Omega]_\times^2 P dm \end{bmatrix} - \begin{bmatrix} dF_{ext} \\ [P]_\times dF_{ext} \end{bmatrix} \right)$

w/ $\int_B dm = m$

$\int_B P dm = 0$ — $S = COG$

$\int_B -[P]_\times[P]_\times dm = \int_B [P]_\times^T[P]_\times dm = \Theta_S$ — moment of inertia

⇒ $0 = \delta W = \begin{pmatrix}\delta r_S \\ \delta\Phi\end{pmatrix}^T \left(\begin{bmatrix} I_{3\times3} m & 0 \\ 0 & \Theta_S \end{bmatrix}\begin{bmatrix} a_S \\ \dot\Psi \end{bmatrix} + \begin{bmatrix} 0 \\ [\Omega]_\times \Theta_S \Omega \end{bmatrix} - \begin{pmatrix} F_{ext} \\ T_{ext} \end{pmatrix} \right)$

$\forall \begin{pmatrix}\delta r_S \\ \delta\Phi\end{pmatrix}$ → ∵ free-body can move in all directions

⇒ means that

$m a_S - F_{ext} = 0$

$\Theta_S \dot\Psi + [\Omega]_\times \Theta_S \Omega - T_{ext} = 0$

note that

$P_S = m V_S$ — momentum

$\dot{P}_S = m a_S$ (f) — change in momentum

$N_S = \Theta_S \Omega_S$ — angular momentum

$\dot{N}_S = \Theta_S\dot\Psi + \Omega \times \Theta_S \Omega$ — change in angular momentum

⇒ $\dot{P}_S = F_{ext}$ — force — newton

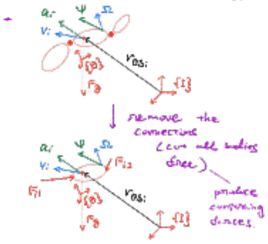
$\dot{N}_S = T_{ext}$ — torque — euler

"change in linear/angular momentum"

△ some reminder

	linear	rotational
θ	x	θ
v	v	ω
a	a	α
motion	$x = vt$ $V = V_0 + at$ $x = v_0 t + \frac{1}{2}at^2$ $v^2 = v_0^2 + 2ax$	$\theta = \omega t$ $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$
mass	m	I moment of inertia
	$F = ma$	$\tau = I\alpha$ torque
	$P = mv$	$L = I\omega$ angular momentum
work	Fd	$\tau\theta$
	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
power	Fv	$\tau\omega$

△ More on Newton-Euler method (when we are not free body)



remove the connections (cut all bodies free) → produce constraining forces

- based on this, w/ n_b rigid bodies
  - $6n_b$ equation
  - $5n_b$ constrained force

一般通过 → 以 generalized coordinates 来表达 — why 5 n_b: if 1 DOF joints, you are impeding 5 DOF constraints in each joint

△ projected Newton-Euler (in generalized motion direction)

- recall

$0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}^T \left(\begin{bmatrix} \dot{P}_{S_i} \\ \dot{N}_{S_i} \end{bmatrix} - \begin{bmatrix} F_{ext,i} \\ T_{ext,i} \end{bmatrix} \right) \quad \forall \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}$

now, this is limited due to linkage!

- now express in generalized coordinates to represent the joints
- express linear/angular momentum in generalized coordinates

$\begin{bmatrix} v_S \\ \Omega \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\dot{q}$

$\begin{bmatrix} a_S \\ \dot\Psi \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\ddot{q} + \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{q}$

⇒ $\begin{bmatrix} \dot{P}_{S_i} \\ \dot{N}_{S_i} \end{bmatrix} = \begin{bmatrix} m a_{S_i} \\ \Theta_{S_i}\dot\Psi_{S_i} + \Omega_{S_i} \times \Theta_{S_i}\Omega_{S_i} \end{bmatrix}$

$= \begin{pmatrix} mJ_{S_i} \\ \Theta_{S_i}J_{R_i} \end{pmatrix}\ddot{q} + \begin{pmatrix} m\dot{J}_{S_i}\dot{q} \\ \Theta_{S_i}\dot{J}_{R_i}\dot{q} + \Omega_{S_i} \times \Theta_{S_i} J_{R_i}\dot{q} \end{pmatrix}$

- virtual displacement in generalized coordinates

$\begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\delta q$ — 不再有刚无化 multi-body 不能无限...

∴ $0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \left(\begin{bmatrix} \dot{P}_{S_i} \\ \dot{N}_{S_i} \end{bmatrix} - \begin{bmatrix} F_{ext,i} \\ T_{ext,i} \end{bmatrix} \right)$

$= \delta q^T \sum_{i=1}^{n_b} \Big\{ \begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} mJ_{S_i} \\ \Theta_{S_i}J_{R_i} \end{pmatrix}\ddot{q}$ — $M(q)$

$+ \begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} m\dot{J}_{S_i}\dot{q} \\ \Theta_{S_i}\dot{J}_{R_i}\dot{q} + \Omega_{S_i} \times \Theta_{S_i} J_{R_i}\dot{q} \end{pmatrix}$ — $b(q,\dot{q})$

$- \begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} F_{ext,i} \\ T_{ext,i} \end{pmatrix} \Big\}$ — $g(q)$

$\forall \delta q$

搞了这么久，就是为了用"q"

- $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = 0$

$M = \sum_{i=1}^{n_b} ({}_A J_{S_i}^T m\, {}_A J_{S_i} + {}_B J_{R_i}^T\, {}_B\Theta_{S_i}\, {}_B J_{R_i})$

$b = \sum_{i=1}^{n_b} ({}_A J_{S_i}^T m\, {}_A\dot{J}_{S_i}\dot{q} + {}_B J_{R_i}^T ({}_B\Theta_{S_i}\, {}_B\dot{J}_{R_i}\dot{q} + {}_B\Omega_{S_i} \times {}_B\Theta_{S_i}\, {}_B J_{R_i}\dot{q}))$

$g = \sum_{i=1}^{n_b} (-{}_A J_{S_i}^T\, {}_A F_{g,i})$

△ Lagrange II

- Lagrangian $\mathcal{L} = T - U$ — kinetic, potential
- Lagrangian Equation

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau$

∵ $U = U(q)$, $T = T(q, \dot{q})$

∴ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = \tau$ — inertial force, gravity vector

△ Kinetic energy

- joint space $T = \frac{1}{2}\dot{q}^T M \dot{q}$
- bodies space

$T = \sum_{i=1}^{n_b}(\frac{1}{2}m_i \dot{r}_{S_i}^T \dot{r}_{S_i} + \frac{1}{2}\Omega_{S_i}^T \Theta_{S_i} \Omega_{S_i})$

w/ $\dot{r}_{S_i} = J_{S_i}\dot{q}$, $\Omega_{S_i} = J_{R_i}\dot{q}$

∴ $T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \left(\sum_{i=1}^{n_b}(J_{S_i}^T m J_{S_i} + J_{R_i}^T \Theta_{S_i} J_{R_i}) \right)\dot{q}$ — $M(q)$

△ potential energy

- gravity

$F_{g_i} = m_i g_I e_g \Rightarrow U_g = -\sum_{i=1}^{n_b} r_{S_i}^T F_{g_i}$

- spring

$F_i = k_i(\|r_i - r_0\| - d_0)\frac{r_i - r_0}{\|r_i - r_0\|}$

$\Rightarrow U_{S_i} = \frac{1}{2}k_i(d(q) - d_0)^2$

△ External Force

- given N_{ext}, F_j, T_k
- $\tau_{F,ext} = \sum_{j=1}^{n_f} J_{P_j}^T F_j$ — task space → joint space
- $\tau_{T,ext} = \sum_{k=1}^{n_m} J_{R,k}^T T_k$

actuator torques

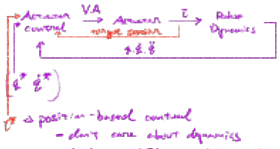
$\tau_{a,k} = (J_{S_k} - J_{S_{k-1}})^T F_{a,k} + (J_{R_k} - J_{R_{k-1}})^T T_{a,k}$

△ Dynamic Control

how are the N-E, how do we end in?

$M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau + J_c^T F_c$

how do we get this



△ position-based control
- don't care about dynamics
- high gain PID → good joint-level tracking
- disturbances are compensated by PID
- cannot control force directly
  ⇒ interaction force can only be controlled w/ compliant surface

△ integrate force-feedback (Dynamic control)
- active regulation of system dynamics
- model-based load compensation
- interaction force control.

△ Joint Impedance Control

- $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau$
- get desired τ

△ torque as function of P/V error

$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$

can think of it as spring force & damping

$\Rightarrow M(q)\ddot{q} + b(q,\dot{q}) + g(q) = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$

static offset due to gravity (when static $M\ddot{q}, b = 0$, $q^* - q \neq 0$)

△ impedance control & gravity compensation

$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q}) + \hat{g}(q)$

↳ configuration dependent, e.g. CoG shifts; estimated

△ independent of configuration — inverse dynamics control

- $\tau = \hat{M}(q)\ddot{q}^* + \hat{b}(q,\dot{q}) + \hat{g}(q)$

get $\ddot{q}^*$, & throw into this EoD, and get the final τ. based on inverse kinematics → inverse dynamics

- assume no dynamic modelling error → result in $\ddot{q} = \ddot{q}^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$

$\omega = \sqrt{k_p}$, $D = \frac{k_d}{2\sqrt{k_p}}$

- describe from task space

$\dot{w}_e = \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix}_e = J_e\ddot{q} + \dot{J}_e\dot{q}$

$\therefore \ddot{q} = J_e^+(\dot{w}_e - \dot{J}_e\dot{q})$

& similarly, multi-task

- $\ddot{q} = \begin{bmatrix} J_1 \\ \vdots \\ J_{n_T} \end{bmatrix}^+ \left(\begin{bmatrix} \dot{w}_1 \\ \vdots \\ \dot{w}_{n_T} \end{bmatrix} - \begin{bmatrix} \dot{J}_1 \\ \vdots \\ \dot{J}_{n_T} \end{bmatrix} \dot{q} \right)$

parallel

- $\ddot{q} = \sum_{i=1}^{n_T} N_i \ddot{q}_i$ w/ $\ddot{q}_i = (J_i N_i)^+ \left(\dot{w}_i^* - \dot{J}_i\dot{q} - J_i \sum_{k=1}^{i-1} N_k \ddot{q}_k\right)$

- get $\ddot{q}$, & insert back to E.O.M.

△ task-space dynamics

- recall joint space

$M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau$

- for end-effector

$\Lambda \dot{w}_e + \mu + p = F_e$

linking the two

$\begin{cases} \tau = J_e^T F_e \\ \dot{w}_e = \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix}_e = J_e\ddot{q} + \dot{J}_e\dot{q} \end{cases}$

$M^{-1}(\tau - b - g)$

derivation

$\Rightarrow \dot{w}_e = J_e M^{-1}(\tau - b - g) + \dot{J}_e\dot{q}$

$\Rightarrow \dot{w}_e - \dot{J}_e\dot{q} + J_e M^{-1} b + J_e M^{-1} g = J_e M^{-1}\tau$

$\Rightarrow \dot{w}_e - \dot{J}_e\dot{q} + J_e M^{-1} b + J_e M^{-1} g = J_e M^{-1} J_e^T F_e$

$\Rightarrow \Lambda \dot{w}_e + \mu + p = F_e$

↳ geometric interpretation: inertial ellipsoid (depend on curr. configuration)

$\therefore$ get $\dot{w}_e^* = k_p E(\chi_e^* - \chi_e) + k_d(w_e^* - w_e) + \dot{w}_e(t)$ → feed-forward

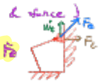
determine desired acceleration → task-level force → Jacobi transformation → joint-space torques

$\tau^* = \hat{J}^T(\hat{\Lambda} \dot{w}_e^* + \hat{\mu} + \hat{p})$

△ interaction control — operation space control

(generalized framework to control motion & force)

$F_c + \Lambda \dot{w}_e + \mu + p = F_e$



△ selection force

$\hat{\Lambda} S_M \dot{w}_e + S_F F_c + \hat{\mu} + \hat{p} = F_e$

$\Rightarrow \tau^* = \hat{J}^T F_e$

$\begin{cases} \Sigma_p = \begin{bmatrix} \sigma_{px} & & \\ & \sigma_{py} & \\ & & \sigma_{pz} \end{bmatrix} & \sigma \in \{0, 1\} \\ \Sigma_F = I_3 - \Sigma_p \end{cases}$

1 → can move; 0 → can apply force

$\begin{cases} \Sigma_R = \begin{bmatrix} \sigma_{rx} & & \\ & \sigma_{ry} & \\ & & \sigma_{rz} \end{bmatrix} \\ \Sigma_{RF} = I_3 - \Sigma_R \end{cases}$

w/ C: transformation between contact local frame & world frame

$\therefore S_M = \begin{bmatrix} C^T \Sigma_p C & 0 \\ 0 & C^T \Sigma_R C \end{bmatrix}$

$S_F = \begin{bmatrix} C^T(I_3 - \Sigma_p)C & 0 \\ 0 & C^T(I_3 - \Sigma_R)C \end{bmatrix}$

$\therefore$ inject back to

$\hat{\Lambda} S_M \dot{w}_e + S_F F_c + \hat{\mu} + \hat{p} = F_e$

△ inverse dynamic as QP

1. classic inverse dynamics

recall

$w = J\dot{q}$

$\dot{w} = J\ddot{q} + \dot{J}\dot{q} \Rightarrow \ddot{q}^* = J^+(\dot{w} - \dot{J}\dot{q})$

$\therefore \tau^* = M(q)\ddot{q}^* + b(q,\dot{q}) + g$

$= M(q)[J^+(\dot{w} - \dot{J}\dot{q})] + b(q,\dot{q}) + g$

2. operation space control

$\tau^* = J_e^T F_e$

$= J_e^T(\Lambda \dot{w}_e^* + \mu + p)$

Not only sol^n

$\therefore \tau^* = J_e^T(\Lambda \dot{w}_e^* + \mu + p) + N(J_e^T)\tau_0$

null space; can change my-self

$N(J_e^T) = (I - J^T(JM^{-1}J^T)^{-1}JM^{-1})$ (w.o. changing desired $\dot{w}$)

Mass, weighted pseudo inverse; proof

$\dot{w} = J\ddot{q} + \dot{J}\dot{q}$

$= JM^{-1}(\tau^* - b - g) + \dot{J}\dot{q}$

$= JM^{-1}(J_e^T(\Lambda \dot{w}_e^* + \mu + p) + (I - J^T(JM^{-1}J^T)^{-1}JM^{-1})\tau_0 - b - g) + \dot{J}\dot{q}$

$= 0$

- we can also see:

$\begin{cases} \tau^* = M(q)\ddot{q}^* + b(q,\dot{q}) + g \\ \tau^* = J_t^T F_e = J_t^T(\Lambda \dot{w}_t^* + \mu + p) \\ \quad = J_t^T((J_t M^{-1} J_t)^{-1} \dot{w} + \mu + p) \end{cases}$

$\Rightarrow \ddot{q} = M^{-1}(\tau^* - b - g)$

$= M^{-1}(J_t^T(J_t M^{-1} J_t)^{-1} \dot{w} + \mu + p) - b - g)$

some sort of mass-matrix weighted pseudo-inverse

3. QP

- $\tau = M\ddot{q} + b + g$
  $J\ddot{q} + \dot{J}\dot{q} = \dot{w}$
  $\min \|\ddot{q}\|$
  $\min \|\tau\|$
  ⟷ classic inverse dynamics

- $[M\ \ -I]\begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix} + b + g = 0$
  $[J_e\ \ 0]\begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix} + \dot{J}\dot{q} = \dot{w}_e^*$

⇓ $x = \begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix}$

multi-task:

$\min_x \left\| \begin{bmatrix} M & -I \\ J_e & 0 \end{bmatrix} x - \begin{bmatrix} -b-g \\ \dot{w}_e^* - \dot{J}\dot{q} \end{bmatrix} \right\|_2$

priority:

$\min_x \|[J_e\ \ 0]x - (\dot{w}_e^* - \dot{J}\dot{q})\|_2$

s.t. $[M\ \ -I]x - (-b - g) = 0$

△ Least-square Optimization

- $Ax = b \Rightarrow x = A^+ b$
  ⇕
  $\min_x \|Ax - b\|_2$
  or
  $\min_x \|x\|_2$ s.t. $Ax = b$

- $A_1 x_1 + A_2 x_2 = b$
  $\Rightarrow [A_1\ A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [A_1\ A_2]^+ b$
  ⇕
  $\min_{x_1,x_2} \left\|[A_1\ A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - b\right\|_2$
  or
  $\min_{x_1,x_2} \left\|\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|_2$ s.t. $[A_1\ A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

- $A_1 x - b_1 = 0$
  $A_2 x - b_2 = 0$
  - Equal priority: $x = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^+ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$
  - Hierarchy

$x = A_1^+ b_1 + N(A_1)x_0$

$\min_x \|A_2 x - b_2\|_2$ s.t. $\|A_1 x - b_1\| = c_1$

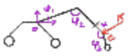
$A_1(A_1^+ b_1 + N(A_1)x_0) = A_1 A_1^+ b_1 + 0 = b_1$

$\Rightarrow A_2 x - b_2 = 0$

$\Rightarrow A_2(A_1^+ b_1 + N(A_1)x_0) - b_2 = 0$

$\Rightarrow x_0 = [A_2 N(A_1)]^+ [b_2 - A_2 A_1^+ b_1]$ — try to fulfill $A_2 x = b_2$

△ case study



- $q = (\varphi_1\ \varphi_2\ x\ \varphi_3)^T$
- $M\ddot{q} + b + g = \tau$
- $J_p \in \mathbb{R}^{2\times4}$
  $J_R \in \mathbb{R}^{1\times4}$

(1) static gravity compensation joint torque?
i.e., ~~$M\ddot{q} + b$~~ $+ g = \tau$

(2) lifting 100 kg, joint torque?

$M\ddot{q} + b + g + J_P^T\begin{pmatrix} 0 \\ 100 \end{pmatrix} = \tau$

$\Rightarrow \tau = g + J_P^T\begin{pmatrix} 0 \\ 100 \end{pmatrix}$

(3) accelerate w/ 1 m/s² upward, φ, x = const.; joint torque?

- $J_t = \begin{bmatrix} J_P \\ J_R \end{bmatrix}$ $\dot{w}_t^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ — might subject to feedback law; my desired acceleration.

- we know that $w = J\dot{q}$, $\dot{w} = J\ddot{q} + \dot{J}\dot{q}$

$\Rightarrow \ddot{q}^* = J^+(\dot{w} - \dot{J}\dot{q}) + N\ddot{q}_0$

$\Rightarrow \tau = M\ddot{q}^* + b + g$

$= MJ^+(\dot{w} - \dot{J}\dot{q}) + b + g$

(1) Task space:

1 m/s² in y, x, φ = const. joint torque? is the sol^n the same?

- $\dot{w}_t^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Lambda = (J_e M^{-1} J_e^T)^{-1}$
  $J_e = J_t = \begin{bmatrix} J_P \\ J_R \end{bmatrix}$ $\mu = \Lambda J_e M^{-1} b - \Lambda \dot{J}_e \dot{q}$
  $p = \Lambda J_e M^{-1} g$

$\Rightarrow \tau^* = J_t^T(\Lambda \dot{w}_t^* + \mu + p)$ (F_e)

(1) interaction control

$\dot{w}_t^* = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ (desired) $J_t = \begin{bmatrix} J_P \\ J_R \end{bmatrix}$

$\tau^* = J_e^T\left[\Lambda \dot{w}_t^* + \mu + p + \begin{bmatrix} -100 \\ -1000 \\ 0 \end{bmatrix}\right]$ — contact force F_c; F_e

(1) QP problems.

(a) least-square minimal torque to get $\dot{w}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

$x = \begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix}$

- $[M\ \ -I]x - (-b - g) = 0$ (1)
  $A_1 x - b_1 = 0$ $\{w = J\dot{q},\ \dot{w} = J\ddot{q} + \dot{J}\dot{q}\}$
- $\begin{bmatrix} J_P & 0 \\ J_R & \end{bmatrix}x - \left(\dot{w}^* - \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{q}\right) = 0$ (2)
  $A_2 x - b_2 = 0$
- $[0\ \ I]x = 0$ (minimize torque, $\min\|\tau\|$)
  $A_3 x - b_3 = 0$ (3)
  as good as possible — then solve.

(b) assume $\ddot{q}_1 = 0$, and $\dot{w}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[M\ \ -I]x - (-b - g) = 0$ (1)

$\begin{bmatrix} J_P & \\ J_R & 0 \\ [1,0,0,0] & \end{bmatrix}x - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \\ [0,0,0,0] \end{bmatrix}\dot{q}\right) = 0$ (2)

(c) assume $\tau_1 = 0$, $\dot{w}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[M\ \ -I]x - (-b - g) = 0$ (1)

$[0\ \ [1,0,0,0]]x = 0$ (2)

$\begin{bmatrix} J_P & 0 \\ J_R & \end{bmatrix}x - \left(\dot{w}^* - \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{q}\right) = 0$ (3)

△ solving a set of QPs

- w/ different priorities
  methods:
  - exploit Nullspace of tasks w/ higher priority
  - every step = QP w/ constraints
  - iteratively nullspace projection
- pseudo-code

```
n_T = number of tasks
x = 0
N_1 = I
for i = 1 → n_T do
    x_i = (A_i N_i)^+ (b_i − A_i x)
    x = x + N_i x_i
    N_{i+1} = N([A_1^T … A_i^T]^T)
end for
```

e.g. @ i = 1

$x = 0$

$N_1 = I$

$x_1 = (A_1 I)^+(b_1 - A\,0)$

$= A_1^+ b_1$

$x := 0 + I A_1^+ b_1$

$N_2 = N(A_1)$

@ i = 2

$x = A_1^+ b_1$

$N_2 = N(A_1)$

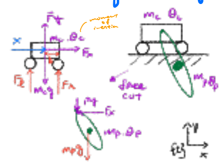
$x_2 = [A_2 N(A_1)]^+[b_2 - A_2 A_1^+ b_1]$

$x := A_1^+ b_1 + N(A_1)[A_2 N(A_1)]^+[b_2 - A_2 A_1^+ b_1]$

→ use solvers (most of the time)

△ Dynamic Modelling Case Study

(1) NE



△ get linear/angular momentum of cart

$m_c \ddot{x}_c = F_x$

$m_c \ddot{y}_c = F_y + F_L + F_R - m_c g$

$\theta_c \ddot{\varphi}_c = F_R b - F_L b$

△ get linear/angular momentum of pendulum

$m_p \ddot{x}_p = -F_x$

$m_p \ddot{y}_p = -F_y - m_p g$

$\theta_p \ddot{\varphi}_p = F_x l\cos(\varphi_p) - F_y l\sin(\varphi_p)$

△ kinematics

$x_c = x$

$y_c = 0$

$\varphi_c = 0$

$[x, \varphi]^T$ ⇒ generalized coordinates.

$x_p = x + l\sin(\varphi)$

$y_p = -l\cos(\varphi)$

$\varphi_p = \varphi$

$\ddot{x}_p = \ddot{x} + \ddot{\varphi}l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi)$

$\ddot{y}_p = \ddot{\varphi}l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi)$

$\ddot{\varphi}_p = \ddot{\varphi}$

- (7),(10-12) in (1) and (4-6)

$m_c\ddot{x} = F_x$ (13)

$m_p\left(\ddot{x} + \ddot{\varphi}l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi)\right) = -F_x$ (14)

$m_p\left(\ddot{\varphi}l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi)\right) = -F_y - m_p g$ (15)

$\theta_p\ddot{\varphi} = F_x l\cos(\varphi) + F_y l\sin(\varphi)$ (16)

- From (13) and (14) remove F_x

$(m_p + m_c)\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - \dot{\varphi}^2 l m_p\sin(\varphi) = 0$

- Insert (13) and (15) in (16) to remove F_x and F_y

$(\theta_p + m_p l^2)\ddot{\varphi} + m_p l\cos(\varphi)\ddot{x} + g l m_p\sin(\varphi) = 0$

(2) NE - projected

$q = \begin{bmatrix} x \\ \varphi \end{bmatrix}$

- $r_{OS_c} = \begin{bmatrix} x \\ 0 \end{bmatrix}$
  $J_{P_c} = \frac{\partial r}{\partial q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  $\dot{J}_{P_c} = \frac{\partial J_{P_c}}{\partial t} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- $r_{OS_p} = \begin{bmatrix} x + l\sin\varphi \\ -l\cos\varphi \end{bmatrix}$
  $J_{P_p} = \frac{\partial r}{\partial q} = \begin{bmatrix} 1 & l\cos(\varphi) \\ 0 & l\sin(\varphi) \end{bmatrix}$
  $\dot{J}_{P_p} = \frac{\partial J_{P_p}}{dt} = \begin{bmatrix} 0 & -l\sin(\varphi)\dot{\varphi} \\ 0 & l\cos(\varphi)\dot{\varphi} \end{bmatrix}$
  $\omega_p = \dot{\varphi}$
  $J_{R_p} = \frac{\partial \omega_p}{\partial \dot{q}} = [0\ \ 1]$

- E.O.M

$\mathbf{M} = \sum \mathbf{J}_{P_i}^T m_i \mathbf{J}_{P_i} + \mathbf{J}_{R_i}^T \Theta_i \mathbf{J}_{R_i} = \mathbf{J}_{P_c}^T m_c \mathbf{J}_{P_c} + \mathbf{J}_{P_p}^T m_p \mathbf{J}_{P_p} + \mathbf{J}_{R_p}^T \theta_p \mathbf{J}_{R_p} = \begin{bmatrix} m_c + m_p & l m_p\cos(\varphi) \\ l m_p\cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}$

$\mathbf{b} = \sum \mathbf{J}_{P_i}^T m_i \dot{\mathbf{J}}_{P_i}\dot{\mathbf{q}} + \mathbf{J}_{R_i}^T \Theta_i \dot{\mathbf{J}}_{R_i}\dot{\mathbf{q}} + \mathbf{J}_{R_i}^T\left(\mathbf{J}_{R_i}\dot{\mathbf{q}}\right) \times \Theta_i \mathbf{J}_{R_i}\dot{\mathbf{q}}$ (=0 (planar system)) $= \mathbf{J}_{P_p}^T m_p \dot{\mathbf{J}}_{P_p}\dot{\mathbf{q}} = \begin{pmatrix} -\dot{\varphi}^2 l m_p\sin(\varphi) \\ 0 \end{pmatrix}$

$\mathbf{g} = \sum_{i=1}^{n} -\mathbf{J}_{S_i}^T \mathbf{F}_i^g = -\mathbf{J}_{P_c}^T\begin{pmatrix} 0 \\ -m_c g \end{pmatrix} - \mathbf{J}_{P_p}^T\begin{pmatrix} 0 \\ -m_p g \end{pmatrix} = \begin{pmatrix} 0 \\ m_p g l\sin(\varphi) \end{pmatrix}$

(3) Lagrange II

- Kinematics cart and pendulum

$\mathbf{r}_{OS_c} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{r}_{OS_p} = \begin{pmatrix} x + l\sin(\varphi) \\ -l\cos(\varphi) \end{pmatrix}$ $\varphi_p = \varphi$

$\dot{\mathbf{r}}_{S_c} = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix}$ $\dot{\mathbf{r}}_{S_p} = \begin{pmatrix} \dot{x} + l\cos(\varphi)\dot{\varphi} \\ l\sin(\varphi)\dot{\varphi} \end{pmatrix}$ $\omega_p = \dot{\varphi}$

- Kinetic and potential energy

$\mathcal{T} = \sum \frac{1}{2}\dot{\mathbf{r}}_{S_i}^T m_i \dot{\mathbf{r}}_{S_i} + \frac{1}{2}\boldsymbol{\omega}_i^T \Theta_i \boldsymbol{\omega}_i = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{x}^2 + \frac{1}{2}m_p l^2\dot{\varphi}^2 + m_p\dot{x}l\cos(\varphi)\dot{\varphi} + \frac{1}{2}\theta\dot{\varphi}^2$

$\mathcal{U} = -m_p g l\cos(\varphi)$ ← 0-level can be chosen

- Equation of motion

$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}} + \frac{\partial \mathcal{U}}{\partial \mathbf{q}} = 0$

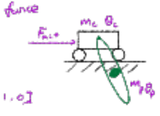
$\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} m_c\dot{x} + m_p\dot{x} + m_p l\cos(\varphi)\dot{\varphi} \\ m_p l^2\dot{\varphi} + m_p\dot{x}l\cos(\varphi) + \theta\dot{\varphi} \end{pmatrix}$

$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} m_c\ddot{x} + m_p\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - m_p l\sin(\varphi)\dot{\varphi}^2 \\ m_p l^2\ddot{\varphi} + m_p\ddot{x}l\cos(\varphi) - m_p\dot{x}l\sin(\varphi)\dot{\varphi} + \theta\ddot{\varphi} \end{pmatrix}$

$-\frac{\partial \mathcal{T}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p\dot{x}l\sin(\varphi)\dot{\varphi} \end{pmatrix}$

$\frac{\partial \mathcal{U}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p g l\sin(\varphi) \end{pmatrix}$

(4) External force



- $J_e^T F_e$

where $J_e = [1, 0]$

$\therefore \tau = J^T F_e$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{act} = \begin{bmatrix} F_{act} \\ 0 \end{bmatrix}$

△ Eigendecomposition