Kinematics

Generalized coordinates

$q = \begin{bmatrix} q_b \\ q_j \end{bmatrix}$ → base, → joints $q_b = \begin{bmatrix} q_{bP} \\ q_{bR} \end{bmatrix} \in SE(3)$

Generalized velocities / accelerations

$u = \begin{bmatrix} {}_I v_B \\ {}_B \omega_{IB} \\ \dot\varphi_1 \\ \vdots \\ \dot\varphi_j \end{bmatrix}$, $\dot u = \begin{bmatrix} {}_I a_B \\ {}_B \psi_{IB} \\ \ddot\varphi_1 \\ \vdots \\ \ddot\varphi_j \end{bmatrix} \in \mathbb{R}^{6+n_j}$

$u = E_{fb}\,\dot q$, $E_{fb} = \begin{bmatrix} I_{3\times3} & & \\ & E_{R\chi} & \\ & & I_{n_j\times n_j} \end{bmatrix}$

P/V of a pt on the robot

- P

${}_I r_{IQ}(q) = {}_I r_{IB}(q) + C_{IB}(q)\,{}_B r_{BQ}(q)$

- V

${}_I v_Q = {}_I v_B + \dot C_{IB}\,{}_B r_{BQ} + C_{IB}\,{}_B \dot r_{BQ}$

$= {}_I v_B + C_{IB}[{}_B\omega_{IB}]_\times {}_B r_{BQ} + C_{IB}\,{}_B \dot r_{BQ}$

$= {}_I v_B - C_{IB}[{}_B r_{BQ}]_\times {}_B\omega_{IB} + C_{IB}\,{}_B \dot r_{BQ}$

$= {}_I v_B - C_{IB}[{}_B r_{BQ}]_\times {}_B\omega_{IB} + C_{IB}\,{}_B J_{q_j}(q_j)\dot q_j$

$\dot r = J_{q_j}\dot q_j$

$= \left[I_{3\times3} \;\; -C_{IB}[{}_B r_{BQ}]_\times \;\; C_{IB}\,{}_B J_{q_j}(q_j)\right] u$

$= {}_I J_Q(q)\, u$

Contact constraints

- C_i : contact point, cannot move

$$\begin{cases} {}_I r_{IC_i} = \text{const} \\ {}_I \dot r_{IC_i} = {}_I J_{C_i} u = 0 \\ {}_I \ddot r_{IC_i} = {}_I J_{C_i}\dot u + {}_I \dot J_{C_i} u = 0 \end{cases}$$

$J_c = \begin{bmatrix} J_{c_1} \\ \vdots \\ J_{c\,n_c} \end{bmatrix} \in \mathbb{R}^{3n_c \times n_n}$ — dim of u; 3× no. of contact pt.

- Nullspace

$0 = \dot r_c = J_c \dot q \Rightarrow \dot q = J_c^+ 0 + N_c \dot q_0 = N_c \dot q_0$

$0 = \ddot r_c = J_c \ddot q + \dot J_c \dot q \Rightarrow \ddot q = J_c^+(-\dot J_c \dot q) + N_c \ddot q_0$

- first priority

task 0: $w_c = J_c \ddot q + \dot J_c \dot q = 0$

task 1: $w_e = J_e \ddot q + \dot J_e \dot q = w_e^*$

- system moved due to contact constraints

Contact Jacobian

- $J_c = \dfrac{\partial r}{\partial q}$ → constraints, → base & joints

$= [J_{c,b} \;\; J_{c,j}]$ — relation between base motion / joint motion & constraints

$= \left[\dfrac{\partial r_c}{\partial q_b} \;\; \dfrac{\partial r_c}{\partial q_j}\right]$

- $\text{rank}(J_{c,b}) \Rightarrow$ base is fully controllable

- $\underbrace{\text{rank}(J_c)}_{\text{total no. of independent constraints}} - \underbrace{\text{rank}(J_{c,b})}_{\text{no. of constraints contributes to motion of the base}}$

⇓

no. of kinematic constraints for joint actuators

e.g. $\begin{cases} q_b = (\,) \\ q_j = (\,) \end{cases}$ d

$J_c = \mathbb{R}^{4\times7}$ x,y constraints of 2 contact point

$\text{rank}(J_c) = 4$

$\text{rank}(J_{c,b}) = 3$

$4 - 3 = 1$ ("1" kinematic constraints of the actuators) "d"

△ generalized coordinates DON'T correspond to the degree of freedom

Dynamics

$M\ddot q + b + g = \tau + J^T F_c$ — contact Jacobian, external force, generalized force

$M(q)\dot u + b(q,u) + g(q) = S^T \tau + J_c^T F_c$ — selection matrix, Jacobian

again

$q = \begin{bmatrix} q_b \\ q_j \end{bmatrix}$ un-actuated base, actuated joints — $u_j = S u = S\begin{bmatrix} u_b \\ u_j \end{bmatrix} = [0, I]\begin{bmatrix} u_b \\ u_j \end{bmatrix}$

that's why "selection matrix"

$\Rightarrow M\ddot q_j + b + g = \tau + J^T F_c$

$\Rightarrow M\ddot q + b + g = S^T\tau + J_c^T F_c$

$\Rightarrow M\dot u + b + g = S^T\tau + J_c^T F_c$ — $S = [0\; I]$; equations in different frames; only joints has torques!

$M\dot u + b + g + J_c^T F_c = S^T \tau$ — (reaction force, contact force, lift)

Soft contact

physical accuracy v. numerical stability

Hard contact

- closed contact

$\dot r^N = 0$

$\ddot r^N = 0$

(open)

- linear complementary constraint

$\dot r^N \ge 0$

$F^N \ge 0$

$\dot r^N F^N = 0$

Hard contact of an MBS

- $M\ddot q + b + g + J_c^T F_c = S^T\tau$ —(1)
- $\dot r_c = J_c \dot q = 0$

$\ddot r_c = J_c \ddot q + \dot J_c \dot q = 0$ —(2)

⇒ from (1)

$\ddot q = M^{-1}(S^T\tau - b - g - J_c^T F_c)$

w/ (2)

$\ddot r_c = J_c\left[M^{-1}(S^T\tau - b - g - J_c^T F_c)\right] + \dot J_c \dot q = 0$ —(3)

- get F_c — contact force from (3)

$F_c = (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}(S^T\tau - (b+g)) + \dot J_c \dot q)$ — given my constraints, actuations, get F_c

∴ w/ (1)

$M\ddot q + b + g + J_c^T F_c = S^T\tau$

$\Rightarrow M\ddot q + b + g + J_c^T(\ldots) = S^T\tau$

$\Rightarrow M\ddot q + b + g + J_c^T(\ldots) = S^T\tau$

define

$N_c = I - M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c$

$J_c N_c = \cancel{J_c} - \cancel{J_c M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c} = 0$

$\Rightarrow N_c^T M\ddot q + N_c^T(b+g) + \underbrace{N_c^T J_c^T}_{=0}(\ldots) = N_c^T S^T \tau$

$\Rightarrow N_c^T M\ddot q + N_c^T(b+g) = N_c^T S^T\tau$

support consistent dynamics — independent of reaction force!

impulse transfer @ contact

$\int_{\{t\}} (M\dot u + b + g + J_c^T F_c - S^T\tau)\,dt$

$= M(u^+ - u^-) + J_c^T \underbrace{\mathcal{F}_c}_{F_c \cdot dt} = 0$ —(1)

also

$\dot r_c^+ = J_c u^+ = 0$ —(2)

⇒ w/ (1)

$u^+ - u^- = -M^{-1} J_c^T \mathcal{F}_c$ —(3)

$\Rightarrow J_c u^+ - J_c u^- = -J_c M^{-1} J_c^T \mathcal{F}_c$

$\Rightarrow J_c u^- = J_c M^{-1} J_c^T \mathcal{F}_c$

$\Rightarrow \dot r_c^- = \Lambda_c^{-1} \mathcal{F}_c$

$\Rightarrow \mathcal{F}_c = \Lambda_c \dot r_c^- = \Lambda_c J_c u^-$ — end effector inertia

∴ $\Delta u = u^+ - u^-$ (from (3))

$= -M^{-1} J_c^T \mathcal{F}_c$

$= -M^{-1} J_c^T \Lambda_c J_c u^-$

$= \underbrace{-M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c u^-}_{\text{change in generalized vel.}}$

∴ $u^+ = u^- - M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c u^-$

$= \left[I - M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c\right] u^-$

$= \underbrace{N_c u^-}_{\text{post-impact vel}}$

N_c also maps $u^- \to u^+$

- $E_{loss} = \Delta E_{kin}$

$= -\frac{1}{2}\Delta u^T M \Delta u$

$= -\frac{1}{2}\Delta \dot r_c^T \Lambda_c \Delta \dot r_c$

$= -\frac{1}{2}\dot r_c^{-T} \Lambda_c \dot r_c^-$

Dynamics control

again

- $M(q)\dot u + b(q,u) + g(q) + J_c^T F_c = S^T\tau$
- $\dot r_c = J_c u$

$\ddot r_c = J_c \dot u + \dot J_c u = 0$

- $F_c = (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}(S^T\tau - b - g) + \dot J_c u)$
- $N_c = I - M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c$
- $N_c^T(M\ddot q + b + g) = N_c^T S^T \tau$

- inverse dynamics

$\tau^* = (N_c^T S^T)^+ N_c^T (M\ddot q + b + g)$

$\tau^* = (N_c^T S^T)^+ N_c^T (M\ddot q + b + g) + \mathcal{N}(N_c^T S^T)\tau_0^*$ — multiple solution; internal force direction

inducing QP

$\min \| A_i x - b_i \|_2$

w/ $x = \begin{bmatrix} \dot u \\ F_c \\ \tau \end{bmatrix}$

(1) $M(q)\dot u + b(q,u) + g(q) + J_c^T F_c = S^T\tau$

$\Rightarrow A_1 = [\hat M \;\; \hat J_c^T \;\; -S^T]$

$b_1 = -\hat b - \hat g$

(2) $J\dot u + \dot J u = \dot w^*$

$\Rightarrow A_2 = [\hat J_i \;\; 0 \;\; 0]$ — motion tasks

$b_2 = \dot w^* - \hat{\dot J}_i u$

(3) $F_i = F_i^*$

$\Rightarrow A_3 = [0 \;\; I \;\; 0]$ — force tasks

$b = F_i^*$

(4) $\min \|\tau\|_2$ — torque min

$\Rightarrow A = [0 \;\; 0 \;\; I]$

$b = 0$

⟶ then solve QP!

case study

(1)

- contact constraints?

△ point on the wheel — point; $q = \begin{bmatrix} x \\ \varphi \end{bmatrix}$ (unactuated base + actuated wheel)

${}_I r_{IP} = \begin{bmatrix} x + r\sin\varphi \\ r + r\cos\varphi \\ 0 \end{bmatrix}$

△ Jacobian

$\dfrac{\partial r}{\partial q} = \begin{bmatrix} \frac{\partial r_1}{\partial x} & \frac{\partial r_1}{\partial \varphi} \\ \frac{\partial r_2}{\partial x} & \frac{\partial r_2}{\partial \varphi} \\ \frac{\partial r_3}{\partial x} & \frac{\partial r_3}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} 1 & r\cos\varphi \\ 0 & -r\sin\varphi \\ 0 & 0 \end{bmatrix}$

△ contact constraints

vel = 0 ⟵ when P point now to the ground. establishing contact

${}_I \dot r_{IP}\big|_{\varphi=\pi} = J\big|_{\varphi=\pi}\dot q$ $\dot r = 0$!

$= \begin{bmatrix} 1 & -r \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot x \\ \dot\varphi \end{bmatrix} = 0$

$\Rightarrow \dot x - r\dot\varphi = 0$ (rolling constraint)

⇒ possible base motion

$\dot q = J_c^+ 0 + N_c \dot q_0$

$= N_c \dot q_0$ ⟶ $N(J_c)$ ∴ $N J_c = 0$

$= \begin{bmatrix} r \\ 1 \end{bmatrix}\dot q_0$

(2)

Floating base system with 12 actuated joint and 6 base coordinates (18DoF)

$J_c = [J_{c,b}\; J_{c,j}]$		1 contact			
total constraints (rank(J_c))	0	3	6	9	12
base constraints (rank($J_{c,b}$))	0	3	5	6	6
internal constraints (rk(J_c) − rk($J_{c,b}$))	0	0	1	3	6
uncontrollable DoF (6 − rk($J_{c,b}$))	6	3	1	0	0

(3) $N_c^T M\ddot q + N_c^T(b+g) = N_c^T S^T \tau$

△ EoM

$\begin{bmatrix} m_0+m_p & 2m_p l\cos\varphi \\ 2m_p l\cos\varphi & 4m_p l^2 + \Theta_p \end{bmatrix}\ddot q + \begin{bmatrix} -2l m_p \dot\varphi^2 \sin\varphi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ m_p g l\sin\varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tau$

△ Contact Jacobian

$J_c = [J_{c,b} \;\; J_{c,j}] = \begin{bmatrix} 1 & 2l\cos\varphi \\ 0 & 2l\sin\varphi \end{bmatrix}$

$q_b = \begin{bmatrix} q_b \\ q_j \end{bmatrix} = \begin{bmatrix} x \\ \varphi \end{bmatrix}$

${}_I r_P = \begin{bmatrix} x + 2l\sin\theta \\ -2l\cos\theta \end{bmatrix}$

$\dfrac{\partial r}{\partial q} = \begin{bmatrix} \frac{\partial r_x}{\partial x} & \frac{\partial r_x}{\partial \varphi} \\ \frac{\partial r_y}{\partial x} & \frac{\partial r_y}{\partial \varphi} \end{bmatrix}$

(a) both constraints active

- $N(J_c) = 0$. no controllable subspace
- contact constraint cannot control the floating base; only change internal force

(b) vertical motion locked

$J_c = [J_{c,b} \;\; J_{c,j}] = [\,0 \;\; 2l\sin\varphi\,]$

$N(J_c) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $N_c^T S^T = [1, 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

- torque has no influence on motion
- only change constraint/reaction/internal force

(c) $J_c = [\,1 \;\; 2l\cos\varphi\,]$

$N_c = \begin{bmatrix} 2l\cos\varphi \\ -1 \end{bmatrix}$

$N_c^T S^T = [2l\cos\varphi \;\; -1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$

- can change the motion
- change the internal force, will change the motion

Control of legged robots

Static/dynamic stability

- Static
  - ≥ 3 legs on ground
  - joints "freeze", will not fall
  - safe, slow, inefficient
- dynamic
  - < 3 legs on ground
  - will fall if no moving
  - hard, fast, efficient

control concepts

- kinematic control
  - high-gain joint position by traj. tracking
- impedance control w/ joint space inverse dynamics
  - low-gain joint control by model compensation
- support-consistent inverse dynamic control
  - projection of dynamics & desired acceleration into Nullspace of contact constraints.
- task-space inverse dynamics control
  - directly regulating in "task-space" as sequential QP

motion planning & control.

motion plan → $q^*, \dot q^*$ → PID → τ → dynamics sys. → $q, \dot q$

get desired $q, \dot q$

pure kinematic, w/ $q, \dot q$

then based on $q, \dot q$, do tracking

BUT: need accurate modelling of environs, to hv good motion planning

otherwise, robot fail!

low gain w/ model compensation

inverse dynamics — lower gains! — feedforward

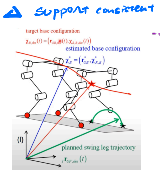
motion plan → $q^*, \dot q^*$ → PID → τ → dynamics sys. → $q, \dot q$

- recall $N_c^T M\ddot q + N_c^T(b+g) = N_c^T S^T\tau$

$\tau = (N_c^T S^T)^+ N_c^T (M\ddot q^* + b + g)$

support consistent dynamics, can better adapt to unknown terrain.

support consistent inverse dynamics



⇒ fully defines the motion of the system

- 12 actuated + 6 actuated = 18 DoF
- 3 swing leg + 6 base motion + 9 contact constraints = 18

$\Rightarrow \ddot q_{des} = \begin{bmatrix} J_F \\ J_B \\ J_C \end{bmatrix}^{-1}\left(\begin{bmatrix} \ddot r_{OF,des} \\ \dot\omega_{B,des} \\ 0 \end{bmatrix} - \begin{bmatrix} \dot J_F \\ \dot J_B \\ \dot J_C \end{bmatrix}\dot q\right)$

more task, then pseudo-inverse, can also hv priority

⇒ after having $\ddot q_{des}$

$\tau = (\hat N_c^T S_c^T)\hat N_c^T(\hat M \ddot q_{des} + \hat b + \hat g)$

QP

$\min_x \| A_i x - b_i \|_2$

$x = \begin{bmatrix} \dot u \\ F_c \\ \tau \end{bmatrix}$

s.t.

- $M(q)\dot u + b(q,u) + g(q) + J_c^T F_c = S^T\tau$

$A = [\hat M \;\; \hat J_c^T \;\; -S^T]$ $b = -\hat b - \hat g$

- $J\dot u + \dot J u = \dot w^*$ (motion task)

$A = [\hat J_i \;\; 0 \;\; 0]$ $b = \dot w^* - \hat{\dot J}_i u$

- $F_i = F_i^*$ (force tasks)

$A = [0 \;\; I \;\; 0]$ $b = F_i^*$

- $\min \|\tau\|_2$ (torque min)

$A = [0 \;\; 0 \;\; I]$ $b = 0$

w/ priority

step 1: move base

try to move the base as good as possible. But not to violate the basics!

$\min_{\ddot q} \| \dot w_{B,des}(\tau) - J_B \ddot q - \dot J_B \dot q \|$

$M\ddot q + b + g + J_c^T F_c = S^T\tau$ EoM

$J_c \ddot q + \dot J_c \dot q = 0$ contact constraints

$F_{c,n,i} > F_{n,min}$ minimal contact force

$\mu F_{c,n,i} \ge \|F_{c,t,i}\|_2$ contact force in friction cone

step 2: move swing leg

try to swing it as good as possible. but not to violate step 1:

$\min_{\ddot q} \| \ddot r_{OF,des}(\tau) - J_F \ddot q - \dot J_F \dot q \|$

$c_1 = \ddot x_{B,des}(\tau) - J_B \ddot q - \dot J_B \dot q$ step 1 not affected

$M\ddot q + b + g + J_c^T F_c = S^T\tau$ EoM

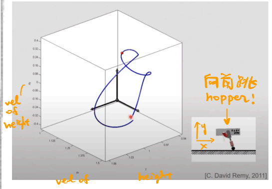
$F_{c,n,i} > F_{n,min}$ minimal contact force

$\mu F_{c,n,i} \ge \|F_{c,t,i}\|_2$ contact force in friction cone

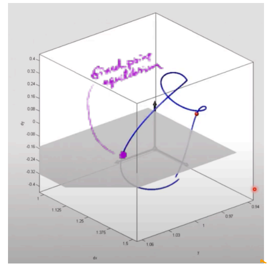
step 3: minimize $\|\tau\|_2$ + $\|F_{c,i}\|$

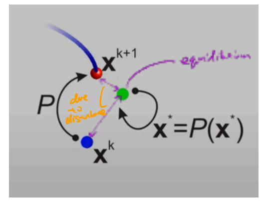
Stability of Legged Robot

- we want dynamic gait!
- limit cycle analysis: look at swing on a step2step basis



vel of x, vel of height, height; 向前跳 hopper!





- Poincaré map $x_{k+1} = P(x_k)$
- fixed point $x^* = P(x^*)$
- linearization $\Delta x_{k+1} = \dfrac{\partial P}{\partial x}\Delta x_k$

$\text{Eig}\left(\dfrac{\partial P}{\partial x}\right) < 1$

⇒ stable

Passive walker efficiency

$\dfrac{E_{used}}{\text{work}} = \dfrac{mgh}{mgd} = \dfrac{h}{d}$

△ ANYmal Case Study

< 10hz Navigation → Motion Planning 10~100 hz (main research)
- motion
- foothold
- reference torque

↓

Tracking ~khz (relatively mature)
- follow the plan
- safety constraints
- reactive behavior

Joint Control ←
- joint torque
- joint position

↓

Plant → proprioception
- joint position
- IMU
- contact sensor

↓ state estimation

exteroception 30hz
- camera → SLAM
- lidar

△ Tracking of ANYmal

- inverse dynamics-based
- whole-body control
- get the "u"

- floating-base model

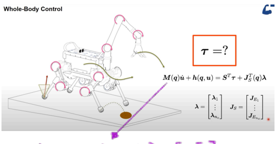
- $q = \begin{bmatrix} {}_I r_{IB} \\ q_{IB} \\ q_j \end{bmatrix}$ generalized coordinates $u = \begin{bmatrix} {}_I v_B \\ {}_B \omega_{IB} \\ \dot{q}_j \end{bmatrix}$ vel

- $M(q)\dot{u} + h(q,u) = S^T\tau + J_S^T(q)\lambda$ external force, ground contacts
  $(b+g)$

$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_c} \end{bmatrix}$

$J_S = \begin{bmatrix} J_{E_1} \\ \vdots \\ J_{E_{n_c}} \end{bmatrix}$

- ${}_S J_{E_k} = [I \quad -C_{IB} \cdot {}_B r^{\times}_{BE_k} \quad 0 \cdots {}_I J_{BE_k} \cdots 0]$



$\begin{bmatrix} M_b \\ M_j \end{bmatrix} \dot{u} + \begin{bmatrix} h_b \\ h_j \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} + \begin{bmatrix} J_{S_b}^T \\ J_{S_j}^T \end{bmatrix} F_S$ (λ)

get τ base on $\dot{u}$ & λ

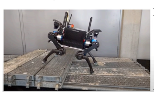
$\Rightarrow \xi = \begin{bmatrix} \dot{u} \\ \lambda \end{bmatrix}$ ⇒ break down into tasks

- EoM
- contact constraints
- force/torque limits

$\begin{cases} W(A\xi - b) = 0 \\ W'(D\xi - f) \le 0 \end{cases}$

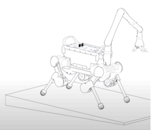
(solve the tasks parallelly / hierarchically
i.e., PROJECT THE CONSTRAINTS INTO THE NULL SPACE OF HIGHER PRIORITY TASKS!)

- examples



Priority	Task
1	Equations of Motion
2	No contact motion
3	Torque limits
4	Friction cone and λ modulation
5	Desired torso x, y position
6	Swing foot motion tracking
7	Limb kinematic limits
8	Main body yaw
9	Main body height
10	Main body roll and pitch
11	Contact force minimization

1	Equations of Motion
1	Force/torque limits
1	No motion at contact points
2	CoM motion tracking
2	Torso angular motion
2	Foot motion tracking
2	End-effector motion tracking
2	End-effector force tracking
2	Torso orientation adaptation
3	Contact force minimisation



- Then after the optimization:
  WE GET ξ, $\xi = \begin{bmatrix} \dot{u} \\ \lambda \end{bmatrix}$
  GET τ!

$\Rightarrow \tau^d = M_j(q)\dot{u}^* + h_j(q,u) - J_{S_j}(q)\lambda^*$

$\Rightarrow$ impedance
$\tau^{cmd} = \tau^d + k_p \tilde{q} + k_D \dot{\tilde{q}}$

△ Planning of ANYmal

- motion
- foothold
- reference torque

} s.t. complex environments

- Traj. Opt / MPC
  - dynamics complexity?
    the "resolution" of the dynamics.
  - integrated/sequential optimization?
    w.r.t. timing/foothold locations

we are converging!
- complex/high-resolution dynamics
- integrated versions of optimization.

- foothold optimization

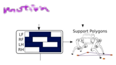
- first optimize the footholds
  i.e., what gait are we going to choose?

$\min_{\xi} \ \frac{1}{2}\xi^T R \xi + c^T \xi$
s.t. $D\xi \le f$

$\xi = [p_{x_1}\ p_{y_1} \ldots p_{x_4}\ p_{y_4} \ldots]$
$\dot{\xi} = f(\xi)$?

- then optimize motion

- w/ gait → polygons of support



- zero-moment point (ZMP)

△ Motion planning + control

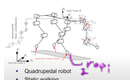
MPC (all in one)

△ Reinforcement Learning

(end 2 end)

△ case study

Kinematics of Floating Base / Mobile Systems



- Quadrupedal robot
- Static walking
- 3 legs in stance [NR 1,2,3]
- 1 in swing [NR 4]

1. How many generalized coordinates?
2. How many base coordinates?
3. How many actuated joint coordinates?
4. How many contact constraints?

1. $12 + 6 = 18$
2. 6
3. 12
4. 9 (@ this frame)

5. Write down the contact constraint
6. How many DoFs remain adjustable?
7. Which DoFs remain adjustable?

5. $\begin{cases} {}_I \dot{r}_{OP_i} = {}_I J_{OP_i} \dot{q} = 0 \\ {}_I \ddot{r}_{OP_i} = {}_I J_{OP_i} \ddot{q} + {}_I \dot{J}_{OP_i} \dot{q} = 0 \\ i = 1,2,3 \end{cases}$

△ ${}_I J_{OP_i} = {}_3[\ \overset{18}{}\ ]$

$9 \begin{bmatrix} {}_3[\quad \overset{18}{} \quad] \\ {}_3[\quad] \\ {}_3[\quad] \end{bmatrix}$

⇒ 9 independent constraints

△

8. Given a desired swing velocity ${}_I \dot{\mathbf{r}}^{des}_{OP_{F4}}$ what is the generalized velocity?
   $\dot{\mathbf{q}} = f\left(\mathbf{q}, {}_I \dot{\mathbf{r}}^{des}_{OP4}\right)$
9. Is it unique?
10. Is it possible to follow the desired swing trajectory without moving the joints of leg 4? How?

8. $\begin{bmatrix} {}_I \dot{r}_{OP_1} = {}_I J_{OP_1} \dot{q} = 0 \\ {}_I \dot{r}_{OP_2} = {}_I J_{OP_2} \dot{q} = 0 \\ {}_I \dot{r}_{OP_3} = {}_I J_{OP_3} \dot{q} = 0 \\ {}_I \dot{r}_{OP_4} = {}_I J_{OP_4} \dot{q} = {}_I \dot{r}^*_{OP_4} \end{bmatrix}$

${}_{12}[\ \overset{18}{}\ ]$

9. NOT UNIQUE

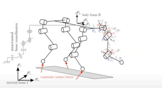
10. yes.

$J_1 = {}_{12}[\overset{18}{J}]$ $w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{r}^* \end{bmatrix}$

$J_2 = {}_3[\overset{18}{0 \ \ J_3}]$ $w_2 [0]$

15 constraints, → not unique

Kinematic Singularity



Task 1:
- contact constraints
- foot motion

Task 2:
- base position
- base orientation

A) Single Task

B) Multiple Tasks

- There exist different formulations for "moving the foot (task1) while keeping the base position and orientation (task2)". Write down the solution for A and B!
- What is the difference?
- What happens in singular configs?

$J_1 = {}_{12}[\overset{18}{J}]$ $w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{r}^* \end{bmatrix}$

$J_2 = 6 \begin{bmatrix} {}_I J_{OB} \\ {}_I J_{ORT} \end{bmatrix}^{18}$ $w_2 = [0]$

(A)
$\dot{q} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}^+ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
↳ if no inverse
⇒ violate all constraints

(B)
$\dot{q} = J_1^+ w_1 + N_1 \dot{q}_0$
$J_2 \dot{q} = J_2(J_1^+ w_1 + N_1 \dot{q}_0) = w_2$
$\dot{q}_0 = (J_2 N_1)^+ [w_2 - J_2 J_1^+ w_1]$

△ summary for floating-base

- EoM
  $M\ddot{q} + b + g + J_c^T F_c = S^T\tau$ —(1)

- contact constraint
  $\dot{r}_c = J_c \dot{q} = 0$
  $\ddot{r}_c = J_c \ddot{q} + \dot{J}_c \dot{q} = 0$ —(2)

- w/ (1) & (2)
  $\ddot{q} = M^{-1}[S^T\tau - (b + g + J_c^T F_c)]$
  $\Rightarrow \ddot{r}_c = J_c M^{-1}[S^T\tau - (b+g+J_c^T F_c)] + \dot{J}_c \dot{q} = 0$

- $J_c M^{-1}[S^T\tau - (b+g+J_c^T F_c)] + \dot{J}_c \dot{q} = 0$ (try to get this)
  $\Rightarrow J_c M^{-1}[S^T\tau - (b+g+J_c^T F_c)] = -\dot{J}_c \dot{q}$
  $\Rightarrow [S^T\tau - (b+g+J_c^T F_c)] = -M J_c^{-1} \dot{J}_c \dot{q}$
  $\Rightarrow J_c^T F_c + b + g - S^T\tau = M J_c^{-1} \dot{J}_c \dot{q}$
  $\Rightarrow J_c^T F_c = M J_c^{-1} \dot{J}_c \dot{q} - b - g + S^T\tau$
  $\Rightarrow F_c = J_c^{-T}\{M J_c^{-1} \dot{J}_c \dot{q} + [S^T\tau - (b+g)]\}$
  $\Rightarrow F_c = J_c^{-T} M J_c^{-1} \dot{J}_c \dot{q} + J_c^{-T} M J_c^{-1} J_c M^{-1}[S^T\tau - (b+g)]$
  $\Rightarrow F_c = [J_c^{-T} M J_c^{-1}][\dot{J}_c \dot{q} + J_c M^{-1}[S^T\tau - (b+g)]]$
  $= (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}[S^T\tau - (b+g)] + \dot{J}_c \dot{q})$

- $F_c = (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}[S^T\tau - (b+g)] + \dot{J}_c \dot{q})$

- recall EoM $M\ddot{q} + b + g + J_c^T F_c = S^T\tau$
  $M\ddot{q} + b + g + J_c^T (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}[S^T\tau - (b+g)] + \dot{J}_c \dot{q}) = S^T\tau$
  $\Rightarrow$ & w/ $N_c = [I - M^{-1} J_c^T [J_c M^{-1} J_c^T]^{-1} J_c)$
  $N_c^T = (I - J_c^T [J_c M^{-1} J_c^T]^{-1} J_c M^{-1})$
  $\Rightarrow$ & $N_c^T J_c^T (J_c M^{-1} J_c^T)^{-1}$
  $\Rightarrow (I - J_c^T [J_c M^{-1} J_c^T]^{-1} J_c M^{-1}) J_c^T (J_c M^{-1} J_c^T)^{-1}$
  $\Rightarrow J_c^T (J_c M^{-1} J_c^T)^{-1} - J_c^T \cancel{[J_c M^{-1} J_c^T]^{-1} J_c M^{-1} J_c^T} (J_c M^{-1} J_c^T)^{-1}$
  $\Rightarrow J_c^T (J_c M^{-1} J_c^T)^{-1} - J_c^T (J_c M^{-1} J_c^T)^{-1} = 0$

- w/ $N_c [J_c^T (J_c M^{-1} J_c^T)^{-1}] \Rightarrow J_c N_c = 0$
  we hv
  $\Rightarrow N_c^T M \ddot{q} + N_c^T (b+g) = N_c^T S^T \tau$
  This is called "support consistent dynamics"
  projector: project to nullspace

△ remarks

- how can we do ID control for fixed base sys?
  $\Rightarrow M\ddot{q} + b + g = \tau$
  $\Rightarrow \tau^* = \hat{M}\ddot{q}^* + \hat{b} + \hat{g}$
  ↳ how the system should move? i.e., desired $\ddot{q}$

- for floating-base systems
  $M\ddot{q} + b + g + J_c^T F_c = S^T\tau$
  - we need to figure out this first.
  - we are constrained in $\ddot{q}$
  - but chicken-&-egg.
    → how to get rid of it?

$0 = J_c \ddot{q}^* + \dot{J}_c \dot{q}$
$\ddot{q}^* = J_c^+(-\dot{J}_c \dot{q})$
$= -J_c^+ \dot{J}_c \dot{q} + N_c \ddot{q}_0$

project into another space
- w/ no. of constraints (no. of forces)
  DoF < (generalized coordinate & DoF)
- project to subspace

THAT IS WHY

$N_c^T M \ddot{q} + N_c^T (b+g) = N_c^T S^T \tau$

$\Rightarrow \tau^* = [N_c^T S^T]^+ N_c^T (M\ddot{q}^* + b + g)$

w/ redundancy — cannot be random, must confined to constraints — usually throw into

$\Rightarrow \tau^* = [N_c^T S^T]^+ N_c^T (M\ddot{q}^* + b + g) + \text{null}[N_c^T S^T]\tau_0$

$\text{null}[N_c^T S^T]$
$= [I - (N_c^T S^T)^+ N_c^T S^T]$

△ getting the contact force

- QR decomposition

EoM
$M\ddot{q} + b + g + J_c^T F_c = S^T\tau$

$\Rightarrow$ let $J_c^T = Q[R^T \ 0]^T$ $[Q^T = Q^{-1}]$

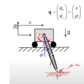
$\Rightarrow Q^T[M\ddot{q} + b + g] + [R^T \ 0]^T F_c = Q^T S^T \tau$

$\Rightarrow \begin{cases} S_c Q^T [M\ddot{q} + b + g] + R F_c = S_c Q^T S^T \tau^* & (1) \\ S_u Q^T [M\ddot{q} + b + g] = S_u Q^T S^T \tau^* & (2) \end{cases}$

order
1. motion planning
   get $\ddot{q}^*$ (confined w/ contact constraints)
2. EoM projected w/ Null-projection
   get τ^* → (2)
3. w/ $\tau^*, \ddot{q}^*$,
   get F_c → (1)

△ case study

- vertical motion locked @ P



soln

- EoM M, b, g: $[\ldots]\ddot{q} + [\ldots] + [\ldots] + J_c^T F_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tau_a$ (S^T)

- motion constraints
  △ $\ddot{y}_P = 0$ $r_{P_y} = -2l\cos\varphi$
  $\Rightarrow J_c = \frac{\partial r}{\partial q} = \begin{bmatrix} \frac{\partial r}{\partial q_b} & \frac{\partial r}{\partial q_j} \end{bmatrix}$
  $= [0 \quad 2l\sin\varphi]$

△ $0 = J_c \ddot{q} + \dot{J}_c \dot{q}$ $N_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\ddot{q} = -J_c^+ \dot{J}_c \dot{q} + (I - J_c^+ J_c)\ddot{q}_0$
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

△ $N_c^T M \ddot{q} + N_c^T (b+g) = N_c^T S^T \tau$
$\Rightarrow N_c^T M(-\cancel{J_c^+ \dot{J}_c \dot{q}} + N_c^T \ddot{q}_0) + \cancel{N_c^T (b+g)} = \cancel{N_c^T S^T} \tau$

$\Rightarrow (m_c + m_p)\ddot{x}_0 = 0 \Rightarrow \ddot{x}_0 = 0$

free moving.
but your current torque does fuck all!

Horizontal lock @ P

$r_{P_x} = x + 2l\sin\varphi$

△ constraints $\ddot{r}_{P_x} = J_c \ddot{q} + \dot{J}_c \dot{q} = 0$
$J_c = [J_{c,b} \ J_{c,j}] = \begin{bmatrix} \frac{\partial r}{\partial q_b} & \frac{\partial r}{\partial q_j} \end{bmatrix}$ $N_c = \begin{bmatrix} 2l\cos\varphi \\ -1 \end{bmatrix}$
$= [1 \quad 2l\cos\varphi]$

△ kinematics
$\ddot{q} = -J_c^+ \dot{J}_c \dot{q} + \underbrace{(I - J_c^+ J_c)}_{N_c}\ddot{q}_0$
$N_c^T S^T = [2l\cos\varphi \ -1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$

△ $N_c^T M(-J_c^+ \dot{J}_c \dot{q} + N_c \ddot{q}_0) + N_c^T (b+g) = -\tau_a$

基於這個 → 得 τ_a

△ no redundancy for τ_a
as $N(N_c^T S^T) = N(-1) = 0$
→ what ever you are doing in τ, affects x & y!

you can change your torque! $N(N_c^T S^T) = N(0) = 1$
→ but does fuck-all!