$u \rightarrow \boxed{\text{plant}} \rightarrow y$

△ n-th order ODE

$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$

$= b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$

T.F. $\frac{Y(s)}{U(s)}$

$= \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$

$\dot{x} = Ax + Bu$

$y = Cx + Du$

△ solution $x(t) = ?$ if $x(t_0) = x_0$

$\dot{x}(t) = Ax(t) + Bu(t)$ —— ①

△ $e^{At}(e^{-At}x(t))'$

$= e^{At}(-A)e^{-At}x(t) + e^{At}e^{-At}\dot{x}(t)$

$= -Ax(t) + \dot{x}(t)$ —— ②

△ given ① & ②

$-Ax(t) + \dot{x}(t) = Bu(t)$

$\therefore e^{At}(e^{-At}x(t))' = Bu(t)$

$(e^{-At}x(t))' = e^{-At}Bu(t)$

△ $\int$ from $t_0 \rightarrow t$

$\int_{t_0}^{t}(e^{-A\tau}x(\tau))'d\tau = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$

$\Rightarrow e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$

$\Rightarrow x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{zero input response}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{zero initial state response}}$

△ 驗算

① $x(t_0)$

$= e^{A(t_0-t_0)}x_0 + \int_{t_0}^{t_0} e^{A(t_0-\tau)}Bu(\tau)d\tau$

$= x_0$ $\quad e^{A(t-\tau)} = e^{At}e^{-A\tau}$

② $\frac{d}{dt}x(t) = \frac{d}{dt}\left[e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(\tau-t)}Bu(\tau)d\tau\right]$

$= A\left[e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\right] + e^{A(t-t_0)}(0 + e^{-A(t-t_0)}Bu(t))$

$= A\left[e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\right] + Bu(t)$

$= Ax(t) + Bu(t)$

△ Definition

$A \in R^{n\times n}$

$e^{At} \triangleq I_n + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^k t^k}{k!} + \cdots$

$= \sum_{j=0}^{\infty} \frac{A^j t^j}{j!}$

△ Equilibrium

$x = x^*$ is an equilibrium of the system 若初 @ x^*, it will remain x^* for the rest

i.e.) $\dot{x} = f(x) = 0$, x^* is the root of $f(x)$

在平衡點附近的系統要趨向於平衡點，但！不代表他是穩定的。

穩的 $\dot{x} @ \theta = \pi$ 時會剛好停在 $\theta = \pi$，不能說是穩定的，因為只要擾動，就會倒下來。假如在系統有一點點

△ Linear

$f: X \rightarrow Y$ is a "linear function"

i.f.f ① additivity $f(x_1+x_2) = f(x_1) + f(x_2)$

② homogeneity $f(\alpha x_1) = \alpha f(x_1)$

in general robotic systems aren't linear.

△ e.g. Euler-Lagrange systems

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$

△ e.g. spring-mass-damper (w/ nonlinear damping)

$ky \quad m\ddot{y} \quad b\dot{y}$

• linear

$m\ddot{y} + c\dot{y} + ky = u$

nonlinear

$m\ddot{y} + c\dot{y} + ky + k_a y^3 = u$

• let $x_1 = y$, $x_2 = \dot{y}$

$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{m}(-cx_2 - kx_1 - ka^2x_1^3) + \frac{1}{m}u \\ y = x_1 \end{cases}$

△ autonomous system not explicitly dependent on t

$\dot{x} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(-cx_2 - kx_1 - ka^2x_1^3) \end{bmatrix}$ $(u=0)$

△ non-autonomous system explicitly dependent on t

$\dot{x} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(-cx_2 - kx_1 - ka^2x_1^3 + \cos\omega t) \end{bmatrix}$

△ nonlinear system

$\dot{x} = f(x, u)$

$y = h(x, u)$

w/ control law

$u = k(x,t)$	$u = k(x)$
$\dot{x} = f(x, k(x,t))$	$\dot{x} = f(x, k(x))$
$y = h(x, k(x,t))$	$y = h(x, k(x))$

△ solution of the nonlinear system

$\dot{x} = f(x,t)$

$x(t_0) = x_0$

$x(t)$?

△ may not have a solution

△ solutions for some initial conditions: not unique

△ solution $\rightarrow \infty$ (may) when $t \rightarrow T$

e.g. non-uniqueness

$\dot{x} = 2\sqrt{x}$, $x(0) = 0$ $\rightarrow x(t) = \begin{cases} 0 & 0 \le t < \infty \\ t^2 & 0 \le t < \infty \end{cases}$

(both solution satisfy) $\dot{x} = 2\sqrt{x}$, $x(0) = 0$

e.g. finite escape time

$\dot{x} = -x + x^2$, $x(0) = x_0$ $\rightarrow x(t) = \frac{x_0}{x_0 + (1-x_0)e^t}$

$\forall t \in \left[\log\frac{x_0}{x_0-1}, \infty\right)$, $x_0 > 1$

$\Rightarrow x_0 + (1-x_0)e^{\log\frac{x_0}{x_0-1}}$

$= x_0 + (1-x_0)\frac{x_0}{x_0-1}$

$= 0$

$\Rightarrow x(t)$ not defined

△ Theorem: Local Existence / Global Uniqueness

$\because \|f(x,t) - f(y,t)\| \le L\|x-y\|$

$\forall x, y \in B = \{x \in R^n \mid \|x - x_0\| \le r\}$ (for local)

$\forall x, y \in R^n$ (for global)

$\forall t \in [t_0, t_1]$

$\therefore \exists\, \delta > 0$ s.t. $\dot{x} = f(x,t)$ w/ $x(t_0) = x_0$ has a unique solution over $[t_0, t_0+\delta]$ — local

$\therefore \dot{x} = f(x,t)$ w/ $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$ — global

global existence & uniqueness for $\dot{x} = Ax$

$\|Ax - Ay\| \le L\|x-y\|$

$\Rightarrow \|Ax - Ay\| \le \|A\|\|x-y\|$ Cauchy-Schwarz

$\Rightarrow$ let $\|A\| = L$

$\therefore \|Ax - Ay\| \le L\|x-y\|$

e.g. multiple equilibrium pts

$\dot{x} = f(x)$, $x(0) = x_0$

def. an equilibrium point is x^* (const.) & $f(x^*) = 0$

nonlinear system have "multiple isolated equilibrium points"

$\dot{x} = -x + x^2$, $x(0) = x_0$

$x^* = 0$ or 1

e.g. limit cycles

explain the periodic behavior (closed loop in the phase space)

• limit cycle is isolated

• limit cycle is not dependent to initial condition

→ some nonlinear systems will converge to a closed curve e.g. Van der Pol oscillator

e.g. bifurcations: different parameters, different equilibrium pts.

e.g. chaos: sensitive to initial condition

△ Controlling nonlinear systems

nonlinear systems:

complexity: peculiar behaviors

difficulty: closed form solutions are unavailable

∴ phase plane methods!

describing functions!

Lyapunov theory!

Phase Plane

• consider a 2-dimensional autonomous sys.

$\dot{x}_1 = f_1(x_1, x_2) \quad x_1(0) = x_{10}$

$\dot{x}_2 = f_2(x_1, x_2) \quad x_2(0) = x_{20}$

$X(t, x_0) = \begin{bmatrix} x_1(t, x_0) \\ x_2(t, x_0) \end{bmatrix}$

def. plane w/ x_1 & x_2 as coordinates is called phase plane

e.g. LC circuit

$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$

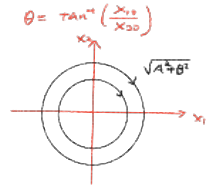
$\begin{bmatrix}x_1(0) \\ x_2(0)\end{bmatrix} = \begin{bmatrix}A \\ B\end{bmatrix}$

sol^n

$x_1(t) = \sqrt{A^2+B^2}\sin(\omega t + \theta)$

$x_2(t) = \sqrt{A^2+B^2}\cos(\omega t + \theta)$

$\theta = \tan^{-1}\left(\frac{x_{10}}{x_{20}}\right)$



slope $= \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$

not singular pts: it is unique if $f_1 \ne 0$ $f_2 \ne 0$, phase trajectories will not intersect

singular pts: it is NOT unique if $f_1 = 0$, $f_2 = 0$, phase trajectories intersect

• For linear systems, the stability is uniquely determined by the nature of their singular points.

• For nonlinear systems, the stability behaviors may be more complex. The system may have multiple isolated singular points some of which are stable and some of which are not. The stable singular point may be globally asymptotically stable or just locally asymptotically stable.

△ Phase Plane Analysis (linear system)

– classification of a singular point

– visualize system trajectory

$\dot{x} = Ax$

$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$

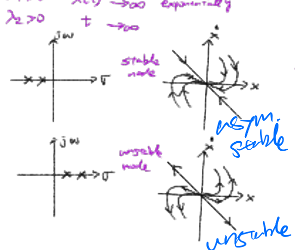
sol^n $\quad x_{1,2}(t) = \begin{cases} k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t} & \lambda_1 \ne \lambda_2 \\ (k_1 + k_2 t)e^{\lambda_1 t} & \lambda_1 = \lambda_2 \end{cases}$

case 1 $\text{sign}(\lambda_1) = \text{sign}(\lambda_2) \in R$

stable/unstable node

△ $\lambda_1 < 0$, $\lambda_2 < 0$: $x(t) \rightarrow 0$ exponentially, $t \rightarrow \infty$

△ $\lambda_1 > 0$, $\lambda_2 > 0$: $x(t) \rightarrow \infty$ exponentially, $t \rightarrow \infty$



case 2. saddle point

$\underset{\in R}{\text{sign}(\lambda_1)} \ne \underset{\in R}{\text{sign}(\lambda_2)}$

assume $\lambda_1 < 0$ $\lambda_2 > 0$

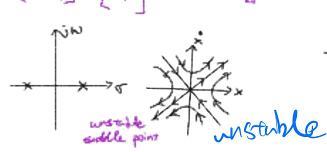
in $x_{1,2}(t) = \begin{cases} k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t}, & \lambda_1 \ne \lambda_2 \\ (k_1 + k_2 t)e^{\lambda_1 t}, & \lambda_1 = \lambda_2 \end{cases}$

△ for $k_2 \ne 0$

$x(t) \rightarrow \infty$, $t \rightarrow \infty$

△ for $k_2 = 0$

$\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} = \begin{bmatrix}k_{11} \\ k_{12}\end{bmatrix}e^{\lambda t} \quad \frac{x_2}{x_1} = \frac{k_{12}}{k_{11}}$



△ phase plane analysis (nonlinear system)

strategy: Jacobian Linearization

recall: $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) & x_1(0) = x_{10} \\ \dot{x}_2 = f_2(x_1, x_2) & x_2(0) = x_{20} \\ x(t, x_0) \end{cases}$

$\Rightarrow$ $\dot{x}_1 = ax_1 + bx_2 + \boxed{g_1(x_1, x_2)}$

$\dot{x}_2 = cx_1 + dx_2 + \boxed{g_2(x_1, x_2)}$ H.O.T.

@ singular pt

$\Rightarrow$ $\dot{x}_1 = ax_1 + bx_2$

$\dot{x}_2 = cx_1 + dx_2$

☞ Remark 2

It can be shown that the phase portrait of (1) in a neighborhood of a singular point is quite similar to the phase portraits of (5) using the Lyapunov theory. For this reason, we call a singular point $\bar{x}^*$ of (1) as a node, saddle point, focus or center if the singular point $\bar{x}^*$ of (5) is the node, saddle point, focus, or center of (5).

△ Jacobian Linearization

$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$

assume $f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$

$x^* = [x_1^*, x_2^*]$ is singular pt.

$A_{x^*} \triangleq \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1, x_2) & \frac{\partial f_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1, x_2) & \frac{\partial f_2}{\partial x_2}(x_1, x_2) \end{bmatrix}\right|_{x = x^*}$

$f_1(x_1, x_2) = f_1(x_1^*, x_2^*) + \frac{\partial f_1}{\partial x_1}(x_1^*, x_2^*)(x_1 - x_1^*) + \frac{\partial f_1}{\partial x_2}(x_1^*, x_2^*)(x_2 - x_2^*) + \mathcal{O}$

$f_2(x_1, x_2) = f_2(x_1^*, x_2^*) + \frac{\partial f_2}{\partial x_1}(x_1^*, x_2^*)(x_1 - x_1^*) + \frac{\partial f_2}{\partial x_2}(x_1^*, x_2^*)(x_2 - x_2^*) + \mathcal{O}$

let $\tilde{x}_1 = x_1 - x_1^*$

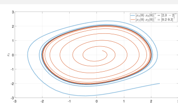
$\tilde{x}_2 = x_2 - x_2^*$

we have

$\begin{cases} \dot{\tilde{x}}_1 = a\tilde{x}_1 + b\tilde{x}_2 + \mathcal{O} \\ \dot{\tilde{x}}_2 = c\tilde{x}_1 + d\tilde{x}_2 + \mathcal{O} \end{cases}$ Jacobian Linearization

△ Limit cycles

❑ Limit Cycles



• The Van der Pol system has an unstable focus at the origin.

• There is a closed curve in the phase portrait. Trajectories inside the curve and those outside the curve all tend to this curve. A motion started on this curve will stay on it forever.

• closed: traj. is closed, periodic nature

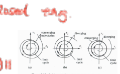
• isolated: traj. is isolated, limiting nature

↳ L is isolated if

$\exists$ region $B = \{x : dist(x, L) < \epsilon, \epsilon > 0\}$

s.t. L is only closed traj. $\in B$

$dist(x, L) = \inf_{y \in L} \|x - y\|$



stable — unstable — semi-stable

case 3 $\lambda_1 \lambda_2 \rightarrow$ complex conjugate w/ non-zero real part

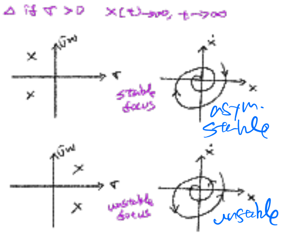
stable/unstable focus

$\lambda_{1,2} = \sigma \pm j\omega$

$\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} = \begin{bmatrix} A\sin(\omega t + \theta) \\ B\cos(\omega t + \theta) \end{bmatrix} e^{\sigma t}$

△ if $\sigma < 0$ $x(t) \rightarrow 0$, $t \rightarrow \infty$

△ if $\sigma > 0$ $x(t) \rightarrow \infty$, $t \rightarrow \infty$



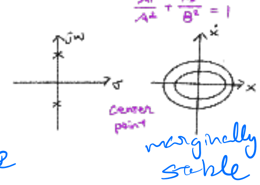
case 4 $\lambda_1 \lambda_2$ complex conjugate w/ zero real part

center point

$\lambda_{1,2} = \pm j\omega$

$\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} = \begin{bmatrix} A\sin(\omega t + \theta) \\ B\cos(\omega t + \theta) \end{bmatrix}$

$\frac{x_1^2}{A^2} + \frac{x_2^2}{B^2} = 1$



△ Existence of Limit Cycles

refer to slides.

康康就要散了.

△ Autonomous System

$\dot{x} = f(x)$

$x \in R^n$, $f(0) = 0$ } W.L.O.G.

→ if $f(x^*) = 0$ & $x^* \neq 0$

then let $z = x - x^*$

$\Rightarrow \dot{z} = \frac{d}{dt}(x - x^*)$

$= \dot{x}$

$= f(x)$

$= f(z + x^*) \triangleq \tilde{f}(z)$

$z = 0$ is the equilibrium point of $\dot{z} = \tilde{f}(z)$

objective: determine the stability of $\dot{x} = f(x)$ w.o. getting the solution

△ Stable

it is said that

「equilibrium pt $x=0$ of $\dot{x} = f(x)$」 (remember the assumption of $f(0)=0$)

is

「stable」

$\begin{cases} \text{if } \forall R > 0 \\ \exists\, r(R) > 0 \end{cases}$ s.t. $\begin{cases} \text{if } \|x(0)\| < r \\ \text{then } \|x(t)\| < R \end{cases}$ $\forall t \geq 0$

o.w. 「unstable」

i.e.,

- $\forall R > 0$, $\exists r > 0$, $\|x(0)\| < r$ $\Rightarrow \forall t \geq 0$ $\|x(t)\| < R$ (the stable condition)
- $\forall R > 0$, $\exists r > 0$, $\|x(0)\| \in B_r$ $\Rightarrow \forall t \geq 0$ $\|x(t)\| \in B_R$

△ Asymptotic Stable

it is said that

「equilibrium point $x=0$ of $\dot{x} = f(x)$」

is

「asymptotically stable」

if

「1. it is stable」

「2. $\exists\, r > 0$ s.t. $\|x(0)\| < r \longrightarrow \lim_{t\to\infty} x(t) = 0$」

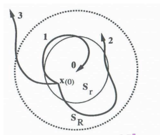
*(convergence condition)

*($B_r = \{x : \|x\| < r\}$ is "domain of attraction")

★

$\dot{x} = Ax$

- is asymptotically stable ⟺ $x=0$ is asymptotically stable in the sense of Lyapunov
- is marginally stable ⟺ $x=0$ is stable in the sense of Lyapunov
- is unstable ⟺ $x=0$ is unstable in the sense of Lyapunov



curve 1 - asymptotically stable
curve 2 - marginally stable
curve 3 - unstable

- Asymptotic stability = Stability + convergence

△ Exponential Stability

$x = 0$ of $\dot{x} = f(x)$

is 「exponentially stable」

if $\exists\, \alpha > 0$, $\lambda > 0$

s.t.

$\|x(t)\| \leq \alpha \|x(0)\| e^{-\lambda t}$ $\forall t \geq 0$

exponential stability

↓

asymptotic stability

↓

stability

△ local/global stability

if asymptotic stability holds for any x_0

the 「equilibrium point」 is

「asymptotically stable in the large」

or

「globally asymptotically stable」

$\dot{x} = Ax$

$\lambda(A)$ has (−) real parts

⇒ origin is globally exponentially stable

⇒ stability for linear system is "global" & "exponential"

△ Lyapunov indirect method

- perform linearization locally
- local stability

△ Jacobian Linearization

- $\dot{x} = \frac{\partial f}{\partial x}\Big|_{x=0} x + \mathcal{O}$

$\triangleq A x + \mathcal{O}$

- $A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

- $A = \frac{\partial f}{\partial x}\Big|_{x=0}$ J of $f(x)$ @ 0

∴ $\dot{x} = Ax$

is the "Jacobian linearization"

of $\dot{x} = f(x)$ @ $x = 0$

△ Lyapunov's Indirect Method

$\dot{x} = Ax \Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

- $\forall \lambda_1, \lambda_2 \ldots \lambda_n$ $Re(\lambda_i) < 0$ then the point is 「locally asymptotically stable」 } Hurwitz
- if any λ_i, $Re(\lambda_i) > 0$ then the point is 「unstable」
- if $\forall \lambda_i$, $Re(\lambda_i) \leq 0$ & $Re(\lambda_i) = 0$ for at least 1, no conclusion

e.g.

- for $\dot{x} = f(x, u)$, $y = h(x)$,

$A = \frac{\partial f}{\partial x}\Big|_{x=0, u=0}$

$B = \frac{\partial f}{\partial u}\Big|_{x=0, u=0}$

$C = \frac{\partial h}{\partial x}\Big|_{x=0}$

$\Rightarrow \begin{cases} \dot{x} = Ax + Bu + \mathcal{O} \\ y = Cx + \mathcal{O} \end{cases}$

$\Rightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ → Jacobian linearization @ $x=0$, $u=0$

- let feedback control law

$u = -Kx$

$\Rightarrow \dot{x} = f(x, -Kx)$ $\quad f(0,0) = 0$

$\triangleq f_c(x)$ $\quad f_c(0) = 0$

$\Rightarrow \frac{\partial f_c(x)}{\partial x}\Big|_{x=0} = \frac{\partial f(x,u)}{\partial x}\Big|_{x=0, u=-Kx}$

$= \frac{\partial f(x,u)}{\partial x}\Big|_{x=0, u=-Kx} + \frac{\partial f(x,u)}{\partial u}\frac{\partial u}{\partial x}\Big|_{x=0, u=-Kx}$

$= A - BK$

⇒ stable i.e. $\forall \lambda(A-BK)_i < 0$

- (A, B) is controllable if

$\text{rank}[B\ AB \cdots A^{n-1}B] = n$

$\exists\, K \in R^{m \times n}$ s.t. $(A - BK)$ is stable

- $\dot{x} = Ax + Bu$ is a small perturbation and valid when x, u are small
- $u = -Kx$ only guarantees "local" asymptotic stability

△ Lyapunov direct method

consider nonlinear spring with damping system

$m\ddot{x} + \underbrace{b\dot{x}|\dot{x}|}_{\text{nonlinear damping}} + \underbrace{k_0 x + k_1 x^3}_{\text{nonlinear spring}} = 0$

⇒ total energy:

$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)\,dx$

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4$

- 「zero energy」 when $\dot{x} = 0$ $x = 0$ equilibrium point
- 「asymptotic stability」 implies mechanical energy ⟶ 0
- 「instability」 implies mechanical energy ⟶ ∞

△ locally positive energy

- $V(x)$ is locally P.D.

if $V(0) = 0$

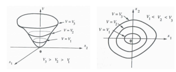
& $V(0) \in B_{R_0}$

i.e., $x \neq 0 \Rightarrow V(x) > 0$

- $V(x)$ is globally P.D.

if $B_{R_0} = R^n$

➢ For $n = 2$



- $V(x_1, x_2)$ typically corresponds to a surface looking like an upward cup.
- The contour curves $V(x_1, x_2) = \text{constant}$ represent a set of ovals surrounding the origin.

△ Lyapunov Function

$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n}\right]$

$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f(x) \leq 0$

⇒ $V(x)$ is a Lyapunov function (local)

if $V(x)$ PD

$\dot{V}(x)$ SND $\quad \forall x \in R^n$

$V(x) \longrightarrow$ global Lyapunov function

△ Lyapunov Stability Theorem (Local)

- $V(x)$ PD, $\dot{V}(x)$ NSD } equilibrium point @ origin is stable
- if $\dot{V}(x)$ is ND } equilibrium point @ origin is asymptotically stable

△ Lyapunov Stability Theorem (Global)

- $V(x)$ PD, $\dot{V}(x)$ ND } @ all point i.e., globally

$V(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$ (radially unbounded)

then equilibrium point @ origin is globally asymptotically stable

△ Invariant Set Theorem

recall:

$\dot{V}(x)$ needs to be ND

⇒ asymptotic stability

↳ Non-trivial in practice!

△ invariant set

a set $G \in R^n$

is an invariant set

if ∀ traj. starting from $p \in G$ remains in G for all time

△ Invariant Set Theorem (local)

$V(x)$

- for some $\ell > 0$, $\Omega_\ell = \{x : V(x) < \ell\}$ is bounded
- $\dot{V}(x) \leq 0$ $\forall x \in \Omega_\ell$
- let R be the set of all pts
  - in Ω_ℓ,
  - $\dot{V}(x) = 0$
- M is the largest invariant set in R (union of all invariant sets)

Then

$x(t)$ originating in Ω_ℓ

tends to M as $t \to \infty$

when $\dot{V} < 0$ i.e., ND

$R = M = \emptyset$

⇒ Lyapunov stability theorem (local) is a special case here

△ Using it

- $V(x)$ is PD
- $\dot{V}(x)$ is NSD
- set R
  - Ω_ℓ (一個 $V(x) < \ell$ 的 ball)
  - $\dot{V}(x) = 0$ (R裡面 $\dot{V}(x) = 0$)
- if set R contains no traj. other than $x \equiv 0$

↳ 「equilibrium point @ origin」

is

「asymptotically stable」

Ω_ℓ is the domain of attraction

△ Invariant Set Theorem (global)

- $V(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$
- $\dot{V}(x) \leq 0$ $\forall x \in R^n$
- set R
  - $\dot{V}(x) = 0$
  - M largest invariant set in R

Then all

$x(t, x_0)$'s that are globally asymptotically

converge to M as $t \to \infty$

易言之:

- 上述條件成立,

任何 $x(t, x_0)$

會 globally asymptotically 收斂至

invariant set M

- 這個 M 勒, 他在 R

i.e., $M \in R$

R 為何物? $R = \{x \mid \dot{V}(x) = 0\}$.

use it?

show that

- $V(x)$ PD
- $\dot{V}(x)$ NSD
- $\dot{V}(x) = 0 \longrightarrow x = 0$.

IN SUM:

- $V(x)$ PD, $\dot{V}(x)$ NSD
- radially unbounded $V(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$
- $\dot{V}(x) = 0$ only when $x = 0$

↳ asymptotically stable

- $V(x)$ PD

$\dot{V}(x)$ NSD

↳ stable